HOMEWORK #9
SOLUTIONS

Problem 9.1

After sampling:

$$\begin{aligned} x[n] &= 7\cos(6000\pi n/9000 + 3\pi/4) + 5\cos(10000\pi n/9000 + 2\pi/3) \\ &= 7\cos(2\pi n/3 + 3\pi/4) + 5\cos(10\pi n/9 + 2\pi/3) \end{aligned}$$

$x[n]$ is the sum of two (discrete-time) sinusoids at frequencies $\hat{\omega}_1 = 2\pi/3$ and $\hat{\omega}_2 = 10\pi/9$. The filter's frequency response is given by: $H(e^{j\hat{\omega}}) = H(z)|_{z=e^{j\hat{\omega}}}$. In this case:

$$\begin{aligned} H(e^{j\hat{\omega}}) &= 6(1 - e^{-j6\hat{\omega}}) = 6e^{-j3\hat{\omega}}(e^{j3\hat{\omega}} - e^{-j3\hat{\omega}}) \\ &= 3je^{-j3\hat{\omega}}\sin 3\hat{\omega} \\ H(e^{j\hat{\omega}_1}) &= 3je^{-j2\pi}\sin 2\pi = 0 \\ H(e^{j\hat{\omega}_2}) &= 3je^{-j10\pi/3}\sin(10\pi/3) = 3je^{j2\pi/3}\sin(-2\pi/3) \\ &= -\frac{3\sqrt{3}}{2}je^{j2\pi/3} \\ |H(e^{j\hat{\omega}_2})| &= \frac{3\sqrt{3}}{2} \\ \angle H(e^{j\hat{\omega}_2}) &= -\pi + \pi/2 + 2\pi/3 = \pi/6 \end{aligned}$$

The output of the filter is:

$$\begin{aligned} y[n] &= (3\sqrt{3}/2)5\cos(10\pi n/9 + 2\pi/3 + \pi/6) \\ &= (15\sqrt{3}/2)\cos(10\pi n/9 + 5\pi/6) \end{aligned}$$

Since $|\hat{\omega}_2| > \pi$, aliasing occurs. The alias frequency is: $10\pi/9 - 2\pi = -8\pi/9$, so the continuous-time output is:

$$y(t) = (15\sqrt{3}/2)\cos(-8\pi 9000t/9 + 5\pi/6) = (15\sqrt{3}/2)\cos(2\pi 4000t - 5\pi/6)$$

Problem 9.2

(a)

$$\begin{aligned} y[n] &= (x[n] - x[n-1])/2 \\ h[n] &= (\delta[n] - \delta[n-1])/2 \\ H(e^{j\hat{\omega}}) &= (1 - e^{-j\hat{\omega}})/2 = je^{-j\hat{\omega}/2}\sin(\hat{\omega}/2) \\ H(z) &= (1 - z^{-1})/2 \end{aligned}$$

(b)

$$\begin{aligned}
h[n] &= (u[n-3]-u[n-10])/7 = \frac{1}{7}\sum_{k=3}^{9}\delta(n-k) \\
y[n] &= h[n]*x[n] = \frac{1}{7}\sum_{k=3}^{9}x[n-k] \\
H(z) &= \frac{1}{7}\sum_{k=3}^{9}z^{-k} = \frac{z^{-3}}{7}\sum_{k=0}^{6}z^{-k} = \frac{z^{-3}}{7}\,\frac{1-z^{-7}}{1-z^{-1}} \\
H(e^{j\hat{\omega}}) &= \frac{e^{-j3\hat{\omega}}}{7}\,\frac{1-e^{-j7\hat{\omega}}}{1-e^{-j\hat{\omega}}} = \frac{e^{-j3\hat{\omega}}}{7}\,\frac{e^{-j7\hat{\omega}/2}}{e^{-j\hat{\omega}/2}}\,\frac{e^{j7\hat{\omega}/2}-e^{-j7\hat{\omega}/2}}{e^{j\hat{\omega}/2}-e^{-j\hat{\omega}/2}} \\
&= \frac{e^{-j6\hat{\omega}}}{7}\,\frac{\sin(7\hat{\omega}/2)}{\sin(\hat{\omega}/2)}
\end{aligned}$$

(c)

$$\begin{aligned}
H(e^{j\hat{\omega}}) &= [7-j\sin 5\hat{\omega}]e^{-j5\hat{\omega}} = 7e^{-j5\hat{\omega}} - j\frac{e^{j5\hat{\omega}}-e^{-j5\hat{\omega}}}{2j}e^{-j5\hat{\omega}} \\
&= 7e^{-j5\hat{\omega}} - \frac{1}{2}(1-e^{-j10\hat{\omega}}) = -\frac{1}{2} + 7e^{-j5\hat{\omega}} + \frac{1}{2}e^{-j10\hat{\omega}} \\
H(z) &= -\frac{1}{2} + 7z^{-5} + \frac{1}{2}z^{-10} \\
h[n] &= -\frac{1}{2}\delta[n] + 7\delta[n-5] + \frac{1}{2}\delta[n-10] \\
y[n] &= h[n]*x[n] = -\frac{1}{2}x[n] + 7x[n-5] + \frac{1}{2}x[n-10]
\end{aligned}$$

Problem 9.3

(a)

$$\begin{aligned}
H(z) &= 100 \\
H(e^{j\hat{\omega}}) &= 100 \\
h[n] &= 100\delta[n] \\
y[n] &= h[n]*x[n] = 100x[n]
\end{aligned}$$

(b)

$$\begin{aligned}
H(z) &= 5 + 6z^{-1} - z^{-2} \\
H(e^{j\hat{\omega}}) &= 5 + 6e^{-j\hat{\omega}} - e^{-j2\hat{\omega}} \\
h[n] &= 5\delta[n] + 6\delta[n-1] - \delta[n-2] \\
y[n] &= h[n]*x[n] = 5x[n] + 6x[n-1] - x[n-2]
\end{aligned}$$

(c)

$$\begin{aligned} H(z) &= (1-z^{-1})[1-2(e^{j2\pi/3}+e^{-j2\pi/3})z^{-1}+4z^{-2}] \\ &= (1-z^{-1})(1-4\cos(2\pi/3)z^{-1}+4z^{-2}) = (1-z^{-1})(1+2z^{-1}+4z^{-2}) \\ &= (1-z^{-1})\frac{1-(2z^{-1})^3}{1-2z^{-1}} \\ H(e^{j\hat{\omega}}) &= (1-e^{-j\hat{\omega}})\frac{1-8e^{-j3\hat{\omega}}}{1-2e^{-j\hat{\omega}}} \end{aligned}$$

To obtain the impulse response it is more convenient to use a different expression for $H(z)$:

$$\begin{aligned} H(z) &= (1-z^{-1})(1+2z^{-1}+4z^{-2}) = 1+z^{-1}+2z^{-2}-4z^{-3} \\ h[n] &= \delta[n]+\delta[n-1]+2\delta[n-2]-4\delta[n-3] \\ y[n] &= h[n]*x[n] = x[n]+x[n-1]+2x[n-2]-4x[n-3] \end{aligned}$$

Problem 9.4

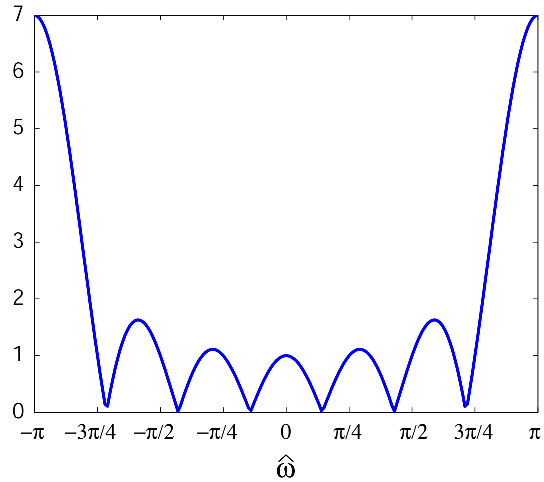
(a)

$$\begin{aligned} h[n] &= \sum_{k=0}^{6}(-1)^k\delta[n-k] \\ H(z) &= \sum_{k=0}^{6}(-1)^k z^{-k} = \frac{1+z^{-7}}{1+z^{-1}} \end{aligned}$$

(b)

$$\begin{aligned} H(e^{j\hat{\omega}}) &= \frac{1+e^{-j7\hat{\omega}}}{1+e^{-j\hat{\omega}}} = \frac{e^{-j7\hat{\omega}/2}}{e^{-j\hat{\omega}/2}}\,\frac{e^{j7\hat{\omega}/2}+e^{-j7\hat{\omega}/2}}{e^{j\hat{\omega}/2}+e^{-j\hat{\omega}/2}} \\ &= e^{-j3\hat{\omega}}\frac{\cos(7\hat{\omega}/2)}{\cos(\hat{\omega}/2)} \end{aligned}$$

(c)



(d) This is a high-pass filter.

Problem 9.5

(a)

$$\begin{aligned} h[n] &= 6\delta[n] - 6\delta[n-1] \\ H(z) &= 6(1 - z^{-6}) \end{aligned}$$

This is the same system function as in Problem 9.1.

(b) $H(e^{j\hat{\omega}}) = 6(1 - e^{-j6\hat{\omega}}) = 12je^{-j3\hat{\omega}} \sin 3\hat{\omega}$

(c) The input signal is:

$$x[n] = 7\cos(2\pi n/3 + 3\pi/4) + 5\cos(10\pi n/9 + 2\pi/3)$$

$x[n]$ is the sum of two discrete-time sinusoids at frequencies $\hat{\omega}_1 = 2\pi/3$ and $\hat{\omega}_2 = 10\pi/9$. In Problem 9.1 it was found that the frequency response at those frequencies is:

$$\begin{aligned} H(e^{j\hat{\omega}_1}) &= 0 \\ H(e^{j\hat{\omega}_2}) &= 3je^{-j10\pi/3}\sin(10\pi/3) = \frac{3\sqrt{3}}{2}e^{j\pi/6} \end{aligned}$$

and that the output of the filter is:

$$y[n] = (15\sqrt{3}/2)\cos(10\pi n/9 + 5\pi/6)$$

The `soundsc` statement implements the D/C converter in the diagram of Problem 9.1, with $f_s = 9000$ samples/sec.. As explained there, aliasing occurs, and the signal produced by `soundsc` is:

$$y(t) = (15\sqrt{3}/2)\cos(2\pi 4000t - 5\pi/6)$$