ECE 2025 - Spring 2005 - HW10 Solutions

Note Title 3/17/2005

10.1) $y[n] = \frac{1}{2} y[n-4] + x[n]$

a) Find impulse response

$$h[n] = \frac{1}{2} h[n-4] + \delta[n]$$

n	$\delta[n]$	$h[n] = \frac{1}{2}h[n-4] + \delta[n]$
0	1	$1 = \frac{1}{2}h[-4] + 1$
1	0	$0 = \frac{1}{2}h[-3] + 0$
2	0	$0 = \frac{1}{2}h[-2] + 0$
3	0	$0 = \frac{1}{2}h[-1] + 0$
4	0	$\frac{1}{2} = \frac{1}{2}h[0] + 0$
5	0	$0 = \frac{1}{2}h[1] + 0$
6	0	$0 = \frac{1}{2}h[2] + 0$
7	0	$0 = \frac{1}{2}h[3] + 0$
8	0	$\frac{1}{2}^2 = \frac{1}{2}h[4] + 0$
⋮	⋮	$\frac{1}{2}^3 = \frac{1}{2}h[8] + 0$
12	0	
⋮	⋮	$\frac{1}{2}^4 = \frac{1}{2}h[12] + 0$
16	0	

$$\therefore \quad h[n] = \begin{cases} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \cdot \delta[n-4k], & n \geq 0 \\ 0, & n < 0 \end{cases}$$

b) response to: $x[n] = n^2[u[n] - u[n-4]]$

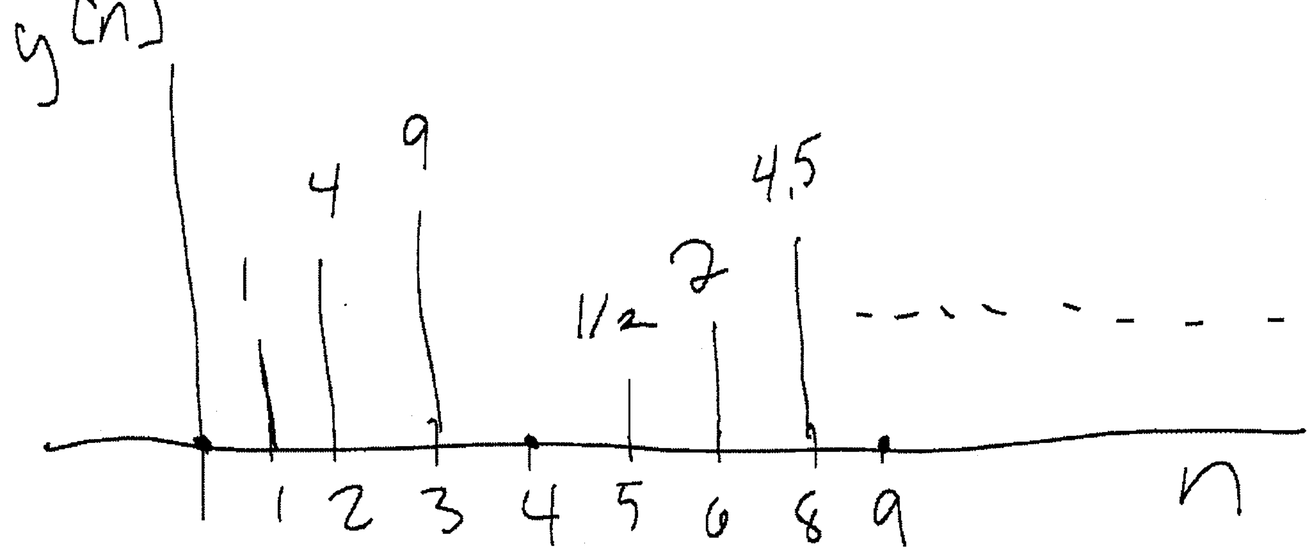
$$x[n] = 0\cdot\delta[n] + \delta[n-1] + 4\delta[n-2] + 9\delta[n-3]$$

if $x[n] = \delta[n]$ then $y[n] = h[n]$

$\therefore\ y[n] = h[n-1] + 4h[n-2] + 9h[n-3]$

$$h[n] = \sum_{k=0}^{\infty}\left(\tfrac{1}{2}\right)^k \delta[n-4k]$$

$$\therefore\ y[n] = \sum_{k=0}^{\infty}\left(\tfrac{1}{2}\right)^k \delta[n-1-4k] + 4\cdot\sum_{k=0}^{\infty}\left(\tfrac{1}{2}\right)^k \delta[n-2-4k] + 9\sum_{k=0}^{\infty}\left(\tfrac{1}{2}\right)^k \delta[n-3-4k]$$



c) y=filter(1, [1,0,0,0,-.5],[0,1,4,9,zeros(1,16)]);
stem(y,'.');

10.2) $$H(z) = \frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}$$

a.) $$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$$

(from transform table on pg. 217)

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

n	$h[n]$
0	1
1	$\frac{1}{2}+1 = 3/2$
2	$\frac{1}{4}+\frac{1}{2} = 3/4$
3	$\frac{1}{8}+\frac{1}{4} = 3/8$
4	$\frac{1}{16}+\frac{1}{8} = 3/16$

b.) $$x[n] = (-1)^n u[n-2]$$

$$x[n] = \sum_{k=2}^{\infty} (-1)^k \delta[n-k]$$

$$y[n] = \sum_{k=2}^{\infty} (-1)^k h[n-k]$$

$$= \sum_{k=2}^{\infty} (-1)^k \left(\frac{1}{2}\right)^{n-k} u[n-k] + \sum_{k=2}^{\infty} (-1)^k \left(\frac{1}{2}\right)^{n-k-1} u[n-k-1]$$

c.) $H(z) = \dfrac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}$

$b = \{1, 1\}$

$a = \{1, -\frac{1}{2}\}$

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y = filter([1,1],[1,0.5],[0,0,(-1).^(2:18)]);
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d.) Find $x[n]$ so $y[n] = 0$ for $n > n_0$

$\Rightarrow$ if $x[n] = \delta[n] - \left(\frac{1}{2}\right)^{n_0} \delta[n-n_0]$

$$
\begin{aligned}
y[n] &= \left(\tfrac{1}{2}\right)^n u[n] + \tfrac{1}{2}^{(n-1)} u[n-1] \\
&\quad - \left(\tfrac{1}{2}\right)^{n_0}\left(\tfrac{1}{2}\right)^{n-n_0} u[n-n_0] - \left(\tfrac{1}{2}\right)^{n_0}\left(\tfrac{1}{2}\right)^{(n-n_0-1)} u[n-n_0-1] \\
&= \left(\left(\tfrac{1}{2}\right)^n u[n] - \left(\tfrac{1}{2}\right)^n u[n-n_0]\right) + \left(\left(\tfrac{1}{2}\right)^{n-1} u[n-1] - \left(\tfrac{1}{2}\right)^{n-1} u[n-n_0-1]\right) \\
&= \tfrac{1}{2}^n \Big(\underbrace{u[n] - u[n-n_0]}_{\text{length} = n_0}\Big) + \tfrac{1}{2}^{n-1} \Big(\underbrace{u[n-1] - u[n-n_0-1]}_{\text{length} = n_0}\Big)
\end{aligned}
$$

$\therefore$ $y[n] = 0$ for $n > n_0$

10.3)

a) $x_a[n] = \left(-\frac{1}{2}\right)^n u[n-5]$

$$= \left(-\frac{1}{2}\right)^5 \cdot \left(-\frac{1}{2}\right)^{n-5} u[n-5]$$

$$X_a(z) = \left(-\frac{1}{2}\right)^5 \cdot \left(\frac{1}{1+\frac{1}{2}z^{-1}}\right) \cdot z^{-5} = -\frac{1}{32} \cdot \frac{z^{-5}}{1+\frac{1}{2}z^{-1}}$$

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b) $x_b[n] = 100\,(0.6)^n u[n] - 100\,(-0.6)^n u[n]$

$$X_b(z) = 100\left(\frac{1}{1-0.6z^{-1}}\right) - 100\left(\frac{1}{1+0.6z^{-1}}\right)$$

$$= \frac{100(1+0.6z^{-1}) - 100\,(1-0.6z^{-1})}{(1-0.6z^{-1})(1+0.6z^{-1})}$$

$$= \frac{120z^{-1}}{(1-0.6z^{-1})(1+0.6z^{-1})} = \frac{120z^{-1}}{1-.36z^{-2}}$$

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c.) $x_c[n] = \delta[n] + u[n-1]$

$$X_c(z) = 1 + \frac{z^{-1}}{1+z^{-1}}$$

$$= \frac{1+z^{-1}}{1+z^{-1}} + \frac{z^{-1}}{1+z^{-1}} = \frac{1+2z^{-1}}{1+z^{-1}}$$

10.4)

$$x[n] \longrightarrow \boxed{H_1(z)} \xrightarrow{v_1[n]} \boxed{H_2(z)} \xrightarrow{v_2[n]} \boxed{H_3(z)} \xrightarrow{y[n]}$$

$$H_1(z) = z^{-1} - z^{-3}$$
$$H_2(z) = 2 + z^{-1}$$

a.) $H_3(z) = ?$

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + a_3 y[n-3] + b_0 v_2[n]$$
$$Y(z) = a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + a_3 z^{-3} Y(z) + b_0 V_2(z)$$
$$Y(z)(1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3}) = b_0 V_2(z)$$
$$H_3(z) = \frac{Y(z)}{V_2(z)} = \frac{b_0}{1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3}}$$

b) $a_1 = a_2 = \frac{1}{2}$; $a_3 = 0$; $b_0 = 7$

$$H_3(z) = \frac{7}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}} = \frac{7}{(1 + \frac{1}{2}z^{-1})(1 - z^{-1})}$$

$$H_1(z) = z^{-1}(1 - z^{-2}) = z^{-1}(1 + z^{-1})(1 - z^{-1})$$

$$H(z) = H_1(z) \cdot H_2(z) \cdot H_3(z)$$
$$= \left(z^{-1}(1+z^{-1})\cancel{(1-z^{-1})}\right) \cdot (2 + z^{-1}) \cdot \frac{7}{(1 + \frac{1}{2}z^{-1})\cancel{(1 - z^{-1})}}$$
$$= \frac{z^{-1}((1+z^{-1})(2+z^{-1}) \cdot 7}{(1 + \frac{1}{2}z^{-1})} = \frac{14z^{-1} + 21z^{-2} + 7z^{-3}}{(1 + \frac{1}{2}z^{-1})}$$

c) $h[n] = 8\delta[n-n_d]$

$H(z) = 8z^{-nd}$

$H(z) = H_1(z) \cdot H_2(z) \cdot H_3(z) = 8z^{-nd}$

$\Rightarrow z^{-1}(1+z^{-1})(1-z^{-1})(2+z^{-1}) \cdot \dfrac{b_0}{1-a_1z^{-1}-a_2z^{-2}-a_3z^{-3}}$

$= z^{-1}(2+z^{-1}-2z^{-2}-z^{-3}) \cdot \dfrac{b_0}{1-a_1z^{-1}-a_2z^{-2}-a_3z^{-3}}$

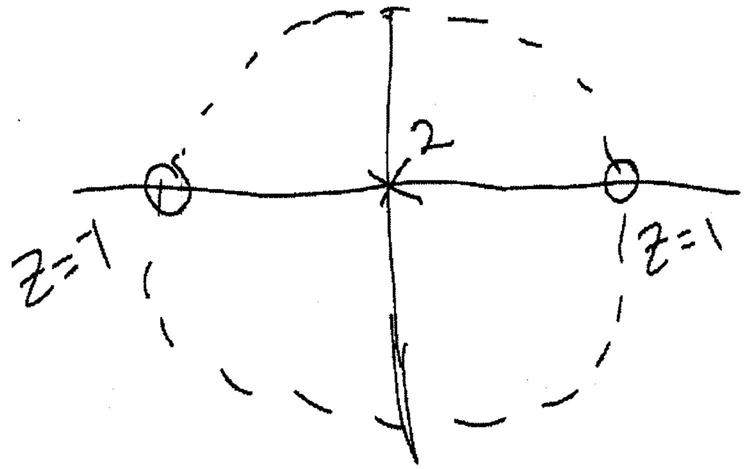
$= 2z^{-1}(1+\frac{1}{2}z^{-1}-z^{-2}-\frac{1}{2}z^{-3}) \cdot \dfrac{b_0}{1-a_1z^{-1}-a_2z^{-2}-a_3z^{-3}}$

let $\boxed{a_1 = -\frac{1}{2},\ a_2 = 1,\ a_3 = \frac{1}{2}}$

$\Rightarrow 2z^{-1} \cdot b_0 = 8z^{-nd}$

$\boxed{\begin{array}{l} b_0 = 4 \\ n_d = 1 \end{array}}$

10.5)



a.)

poles: $z=0$ (2 poles)

zeros: $z=1,-1$

$$H(z) = \frac{(z-1)(z+1)}{z^2} = \frac{z^2-1}{z^2} = 1 - z^{-2}$$

$$z = e^{j\hat{w}} \Rightarrow H(e^{j\hat{w}}) = 1 - e^{-j\hat{w}2}$$

b.) $H(e^{j\hat{w}}) = 1 - e^{-j\hat{w}2}$

$$= e^{-j\hat{w}}\left(e^{j\hat{w}} - e^{-j\hat{w}}\right)$$

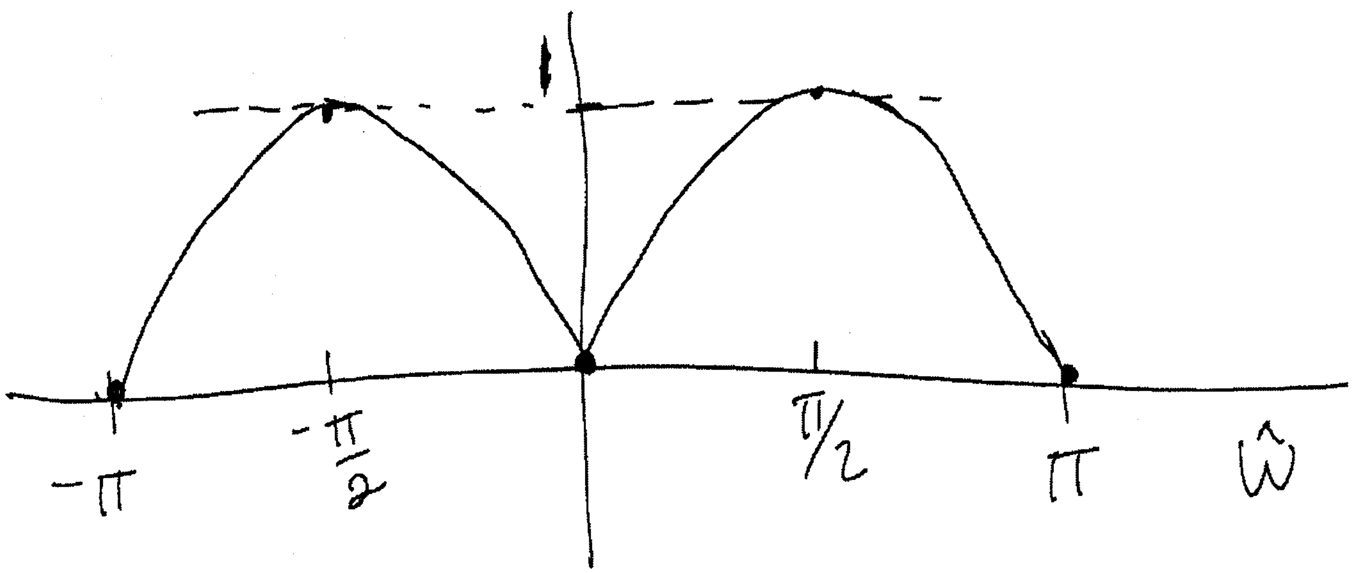
$$= e^{j\hat{w}}(2j \sin \hat{w}) = e^{j(\hat{w}+\pi/2)}(2 \sin \hat{w})$$

$|H(e^{j\hat{w}})| = |2 \sin \hat{w}|$ max value for sin at $\hat{w} = \pm \frac{\pi}{2}$

c) $|H(e^{j\hat{\omega}})| = |2\sin(\hat{\omega})|$

max value $= 2 \quad \therefore \beta = \frac{1}{2}$

plot $\beta \cdot |H(e^{j\hat{\omega}})| = \frac{1}{2}|2\sin(\hat{\omega})| = |\sin(\hat{\omega})|$



PROBLEM 11.6. From the difference equation, the system function is:

$$H(z) = \frac{1+z^{-2}}{1-0.8z^{-1}} = \frac{1}{1-0.8z^{-1}} + \frac{z^{-2}}{1-0.8z^{-1}} = H_1(z) + H_2(z).$$

(a) We will use approach #2 to find the step response for each of the above terms separately, and then add the results to get the overall step response. The Z-transform of the unit step is $\frac{1}{1-z^{-1}}$, so the Z-transform of the step response to the first system is:

$$Y_1(z) = \frac{1}{(1-0.8z^{-1})(1-z^{-1})}$$

$$= \frac{A}{1-0.8z^{-1}} + \frac{B}{1-z^{-1}}$$

$$= \frac{(A+B)-(A+0.8B)z^{-1}}{(1-0.8z^{-1})(1-z^{-1})} \qquad \Longrightarrow A = -4,\ B = 5.$$

$$= \frac{-4}{1-0.8z^{-1}} + \frac{5}{1-z^{-1}} \qquad \longleftrightarrow \qquad y_1[n] = (5 - 4(0.8)^n)u[n].$$

The step response of the second system $H_2(z)$ will be the same, but delayed by two. Therefore, the overall step response is:

$$y[n] = y_1[n] + y_1[n-2] = (5 - 4(0.8)^n)u[n] + (5 - 4(0.8)^{n-2})u[n-2].$$

(b) Use approach #3. The response of an LTI system to an input of $\cos(\hat{\omega}_1 n + \theta)$ is $A\cos(\hat{\omega}_1 n + \theta + \phi)$, where A and ϕ satisfy $Ae^{j\phi} = H(e^{j\hat{\omega}_1})$. But:

$$H(e^{j0.5\pi}) = \left.\frac{1+z^{-2}}{1-0.8z^{-1}}\right|_{z=j} = \frac{1-1}{1+0.8j} = 0.$$

$$H(e^{j0.25\pi}) = \left.\frac{1+z^{-2}}{1-0.8z^{-1}}\right|_{z=e^{j0.25\pi}} = \frac{1-j}{1-0.8e^{-j0.25\pi}} \approx 1.983e^{-j0.542\pi}.$$

Hence, the response to

$$x[n] = 2\cos(0.5\pi n - \pi/2) + \cos(0.25\pi n - \pi)$$

is

$$y[n] = 0 + 1.983\cos(0.25\pi n - \pi - 0.542\pi)$$
$$= 1.983\cos(0.25\pi n + 0.458\pi).$$

(c) In terms of the impulse response $h[n]$, the response to $10\delta[n-5]$ will be $10h[n-5]$. From part (a), the impulse response is:

$$\frac{1}{1-0.8z^{-1}} + \frac{z^{-2}}{1-0.8z^{-1}} \longleftrightarrow \quad h[n] = (0.8)^n u[n] + (0.8)^{n-2}u[n-2]$$

$$= \delta[n] + 0.8\delta[n-1] + (1 + (0.8)^{-2})(0.8)^n u[n-2]$$

$$= \delta[n] + 0.8\delta[n-1] + 2.5625(0.8)^n u[n-2].$$

Therefore, $y[n] = 10h[n-5] = 10\delta[n-5] + 8\delta[n-6] + 25.625(0.8)^{n-5}u[n-7]$.

(d) Use approach #1.