ECE 2025, Spring 2005, Problem Set #11

- Problem 11.1
$$H(z) = \frac{z^{-2}}{4 + 9z^{-2}}$$

(a) Find the impulse response.

Let us first examine a 2nd-order IIR system

$$\begin{aligned} G(z) &= \frac{1}{1 + a^2 z^{-2}} = \frac{1}{1 - ajz^{-1}} \frac{1}{1 + ajz^{-1}} \\ &= \frac{0.5}{1 - ajz^{-1}} + \frac{0.5}{1 + ajz^{-1}}. \end{aligned}$$

The corresponding impulse response is

$$\begin{aligned} g[n] &= 0.5\ (aj)^n u[n] + 0.5(-aj)^n u[n] \\ &= 0.5a^n \left(e^{j0.5\pi n} + e^{-j0.5\pi n}\right) u[n] \\ &= a^n \cos(0.5\pi n) u[n]. \end{aligned}$$

Since
$$H(z) = \frac{z^{-2}}{4}\ \frac{1}{1 + \frac{9}{4} z^{-2}}$$
we infer that $h[n] = 0.25\ g[n-2]$, where $g[n] = a^n \cos(0.5\pi n) u[n]$, $a = 1.5$.

Thus, $h[n] = 0.25 \times 1.5^{n-2} \cos(0.5\pi(n-2)) u[n-2]$.

(b) The system is not stable because $h[n]$ tends to infinity as $n \to \infty$. This is expected since the poles are at $z = \pm 1.5j$ which are outside of the unit circle.

Prob 11.1 *(solution without partial fractions)*

A causal LTI system has the following system function:

$$H(z) = \frac{z^{-2}}{4 + 9z^{-2}}$$

(a) Determine the impulse response of this system. Give a simple (real-valued) expression for your answer.

First of all, convert the system function $H(z)$ to a difference equation:

$$H(z) = \frac{(1/4)z^{-2}}{1 + (9/4)z^{-2}} \qquad \Longrightarrow \qquad y[n] = -(9/4)y[n-2] + (1/4)x[n-2]$$

The impulse response is obtained when $x[n] = \delta[n]$, so the input doesn't "start" until $n = 2$. The *at rest* condition means that we assume $y[n] = 0$ for $n < 2$. With that initial condition, we can "recurse" the difference equation:

$$\begin{aligned}
y[2] &= -(9/4)y[0] + (1/4)x[0] = -(9/4)(0) + (1/4)(1) = 1/4 \\
y[3] &= -(9/4)y[1] + (1/4)x[1] = -(9/4)(0) + (1/4)(0) = 0 \\
y[4] &= -(9/4)y[2] + (1/4)x[2] = -(9/4)(1/4) + (1/4)(0) = -9/16 \\
y[5] &= -(9/4)y[3] + (1/4)x[3] = 0 \\
y[6] &= -(9/4)y[4] + (1/4)x[4] = -(9/4)(-9/16) + (1/4)(0) = 81/64 \\
y[7] &= -(9/4)y[5] + (1/4)x[5] = 0
\end{aligned}$$

Thus for odd indices, $y[n]$ will be zero; and for even indices the formula is a geometric sequence with a ratio between terms of $-9/4$. The general formula is

$$h[n] = \begin{cases} 0 & \text{for } n \text{ odd, or } n < 2 \\ (1/4)(-9/4)^{(n-2)/2} & \text{for } n \geq 2 \text{ and } n \text{ even} \end{cases}$$

Recall that we use $h[n]$ to denote the *impulse response.*

(b) Determine whether or not the system is *stable*. Justify your answer.

The system is *unstable*, because part (a) gives and input-output pair in which the input is bounded but the output is not. The input $x[n] = \delta[n]$ satisfies $|x[n]| \leq 1$ for all n, but as $n \to \infty$, the size of the output grows without bound, i.e., $|y[n]|$ behaves like $(2.25)^n$ which blows up.

- Problem 11.2

$$H(z) = \frac{z^{-1}}{15 + 7z^{-2}}$$

(a) Determine a simple (real-valued) expression for $|H(e^{j\hat{\omega}})|^2$.

From the $H(z)$ expression, we find

$$H(e^{j\hat{\omega}}) = \frac{e^{-j\hat{\omega}}}{15 + 7e^{-2j\hat{\omega}}}$$

Thus,

$$\begin{aligned} |H(e^{j\hat{\omega}})|^2 &= H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}}) \\ &= \frac{e^{-j\hat{\omega}}}{15 + 7e^{-2j\hat{\omega}}} \, \frac{e^{j\hat{\omega}}}{15 + 7e^{2j\hat{\omega}}} \\ &= \frac{1}{15^2 + 7^2 + 2 \times 15 \times 7 \cos(2\hat{\omega})} \\ &= \frac{1}{272 + 210 \cos(2\hat{\omega})}. \end{aligned}$$

(b) Determine the output $y_1[n]$ when the input is

$$x_1[n] = 200 \cos(0.5\pi n).$$

The frequency response at $\hat{\omega} = 0.5\pi$ is:

$$H(e^{j0.5\pi}) = H(z)|_{z=e^{j0.5\pi}=j} = \frac{j^{-1}}{15 - 7j^{-2}} = -\frac{j}{8} = 0.125e^{-j\pi/2}.$$

Therefore, the output $y_1[n] = 25 \cos(0.5\pi n - 0.5\pi)$.

- Problem 11.3

  S_1:

$$y[n] = 0.4y[n-1] + x[n] + x[n-1]$$

We find:

$$H(z) = \frac{1 + z^{-1}}{1 - 0.4z^{-1}}$$

Pole: $z = 0.4$, zero: $z = -1$. No match.

S_2:

$$y[n] = 0.75y[n-1] + x[n] + x[n-1]$$

We find:

$$H(z) = \frac{1 + z^{-1}}{1 - 0.75z^{-1}}$$

Pole: $z = 0.75$, zero: $z = -1$. Matches plot #4.

S_3:

$$y[n] = -0.75y[n-1] + x[n] - x[n-1]$$

We find:

$$H(z) = \frac{1 - z^{-1}}{1 + 0.75z^{-1}}$$

Pole: $z = -0.75$, zero: $z = 1$. Matches plot #2.

S_4:

$$y[n] = 0.75y[n-1] + x[n] - x[n-1]$$

We find:

$$H(z) = \frac{1 - z^{-1}}{1 - 0.75z^{-1}}$$

Pole: $z = 0.75$, zero: $z = 1$. No match.

S_5:

$$y[n] = x[n] - x[n-1] + x[n-2]$$

We find:

$$H(z) = 1 - z^{-1} + z^{-2} = \frac{z^2 - z + 1}{z^2}$$

Poles: $z = 0$ (double), zeros: $e^{\pm j\pi/3}$. No match.

S_6:

$$y[n] = x[n] - x[n-1] + x[n-2] - x[n-3]$$

We find:

$$H(z) = 1 - z^{-1} + z^{-2} - z^{-3} = \frac{z^3 - z^2 + z - 1}{z^3}$$

Poles: $z = 0$ (triple), zeros: $1, \pm j$. Matches plot #3.

S_7:

$$y[n] = x[n] + \frac{1}{4}x[n-1] - \frac{3}{4}x[n-2]$$

We find:

$$H(z) = 1 + \frac{1}{4}z^{-1} - \frac{3}{4}z^{-2} = \frac{z^2 + \frac{1}{4}z - \frac{3}{4}}{z^2}$$

Poles: $z = 0$ (double), zeros: $-1, 0.75$. Matches plot #1.

S_8:

$$y[n] = \frac{1}{3}x[n] - x[n-1] + x[n-2] - \frac{1}{3}x[n-3]$$

We find:

$$H(z) = \frac{1}{3} - z^{-1} + z^{-2} - \frac{1}{3}z^{-3} = \frac{\frac{1}{3}z^3 - z^2 + z - \frac{1}{3}}{z^3}$$

Poles: $z = 0$ (triple), zeros: $z = 1$ (triple). No match.

In summary, #1 = S_7; #2 = S_3; #3 = S_6; #4 = S_2.

- Problem 11.4.

  Recall that

$$a^n u[n] \quad \xrightarrow{z} \quad \frac{1}{1 - az^{-1}}$$

S_1:

$$y[n] = 0.4y[n-1] + x[n] + x[n-1]$$

We find:

$$H(z) = \frac{1 + z^{-1}}{1 - 0.4z^{-1}}$$

$$h[n] = 0.4^n u[n] + 0.4^{n-1} u[n-1].$$

Thus, $h[n]$ is IIR and $h[1] = 0.4 + 1 = 1.4$. No match.

S_2:

$$H(z) = \frac{1 + z^{-1}}{1 - 0.75z^{-1}}$$

We find:

$$h[n] = 0.75^n u[n] + 0.75^{n-1} u[n-1].$$

Thus, $h[n]$ is IIR and $h[1] = 0.75 + 1 = 1.75$. Matches with Figure O.

S_3:

$$y[n] = -0.75y[n-1] + x[n] - x[n-1]$$

We find:

$$H(z) = \frac{1 - z^{-1}}{1 + 0.75z^{-1}}$$

$h[n]$ is IIR. Due to the pole at $z = -0.75$, $h[n]$ oscillates. Thus, the matching plot is Figure M.

S_4:

$$H(z) = \frac{1 - z^{-1}}{1 - 0.75z^{-1}}$$

We find:

$$h[n] = 0.75^n u[n] - 0.75^{n-1} u[n-1].$$

$h[n]$ is IIR and $h[1] = 0.75 - 1 = -0.25$. Thus, the matching plot is Figure K.

S_5:

$$y[n] = x[n] - x[n-1] + x[n-2]$$

We find:

$$h[n] = \delta[n] - \delta[n-1] + \delta[n-2]$$

There is not a matching plot.

S_6:

$$H(z) = 1 - z^{-1} + z^{-2} - z^{-3}$$

We find:

$$h[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3]$$

The matching plot is Figure N.

S_7:

$$y[n] = x[n] + \frac{1}{4}x[n-1] - \frac{3}{4}x[n-2]$$

We find:

$$h[n] = \delta[n] + \frac{1}{4}\delta[n-1] - \frac{3}{4}\delta[n-2]$$

The matching plot is Figure L.

S_8:

$$H(z) = \frac{1}{3}(1 - z^{-1})^3$$

We find:

$$H(z) = \frac{1}{3} - z^{-1} + z^{-2} - \frac{1}{3}z^{-3}$$

$$h[n] = \frac{1}{3}\delta[n] - \delta[n-1] + \delta[n-2] - \frac{1}{3}\delta[n-3]$$

The matching plot is Figure J.

In summary, J=S_8; K=S_4; L=S_7; M=S_3; N=S_6; O=S_2.

- Problem 11.5

  Recall that pole at $z = r\ e^{j\theta} \quad \Rightarrow \quad |H(e^{j\hat{\omega}})|$ peaks at $\hat{\omega} = \theta$.

  Zero of $H(z)$ at $z = r\ e^{j\theta} \quad \Rightarrow \quad |H(e^{j\hat{\omega}})|$ dips at $\hat{\omega} = \theta$.

  S_1:

  $$y[n] = 0.4y[n-1] + x[n] + x[n-1]$$

  We find:

  $$H(z) = \frac{1 + z^{-1}}{1 - 0.4z^{-1}}$$

  The pole is at $z = 0.4$; the zero is at $z = -1$. Moreover, $H(e^{j0}) = H(1) = 2/0.6 = 3.3333$. Thus, the match is Figure D.

  S_2:

  $$H(z) = \frac{1 + z^{-1}}{1 - 0.75z^{-1}}$$

  The pole is at $z = 0.75$; the zero is at $z = -1$. Moreover, $H(e^{j0}) = H(1) = 2/0.25 = 8$. Thus, the match is Figure B.

  S_3:

  $$y[n] = -0.75y[n-1] + x[n] - x[n-1]$$

  We find:

  $$H(z) = \frac{1 - z^{-1}}{1 + 0.75z^{-1}}$$

  The pole is at $z = -0.75$; the zero is at $z = 1$. Moreover, $H(e^{j\pi}) = H(-1) = 2/0.25 = 8$. There is not a match.

  S_4:

  $$H(z) = \frac{1 - z^{-1}}{1 - 0.75z^{-1}}$$

  The pole is at $z = 0.75$; the zero is at $z = 1$. Moreover, $H(e^{j\pi}) = H(-1) = 2/1.75 = 1.1429$. Thus, the match is Figure A.

  S_5:

  $$y[n] = x[n] - x[n-1] + x[n-2]$$

We find:

$$H(z) = 1 - z^{-1} + z^{-2}.$$

The zeros are at $z = e^{\pm j\pi/3}$. Thus, the match is Figure C.

S_6:

$$H(z) = 1 - z^{-1} + z^{-2} - z^{-3}$$

The zeros are at 1, $\pm j$. Thus, the match is Figure E.

S_7:

$$y[n] = x[n] + \frac{1}{4}x[n-1] - \frac{3}{4}x[n-2]$$

We find:

$$H(z) = 1 + \frac{1}{4}z^{-1} - \frac{3}{4}z^{-2}$$

The zeros are at -1, 0.75. Thus, the match is Figure F.

S_8:

$$H(z) = \frac{1}{3}(1 - z^{-1})^3$$

There are three zeros at $z = 1$. Moreover, $H(e^{j\pi}) = H(-1) = 8/3 = 2.6667$. There is not a match.

In summary, A=S_4; B=S_2; C=S_5; D=S_1; E=S_6; F=S_7.