HOMEWORK #14
SOLUTIONS

Problem 14.1

(a)

$$x(t) = \frac{1}{3}u(t)u(3-t) = \begin{cases} 0 & t < 0 \\ 1/3 & 0 \le t \le 3 \\ 0 & t > 3 \end{cases}$$

We can write: $x(t) = \frac{1}{3}y(t - 3/2)$, with: $y(t) = u(t + 3/2) - u(t - 3/2)$. Then:

$$X(j\omega) = \frac{1}{3}e^{-j3\omega/2}Y(j\omega) = e^{-j3\omega/2}\frac{\sin(3\omega/2)}{3\omega/2}$$

(b)

$$\begin{aligned} y(t) &= e^{-7t}u(t-5) = e^{-35}e^{-7(t-5)}u(t-5) \\ Y(j\omega) &= e^{-35}\frac{e^{-j5\omega}}{7+j\omega} \\ X(j\omega) &= j\omega Y(j\omega) = j\omega\frac{e^{-(35+j5\omega)}}{7+j\omega} \end{aligned}$$

(c) Using the following trigonometric identity:

$$\begin{aligned} \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \\ x(t) &= \frac{1}{20}\frac{\sin(15\pi t)}{\pi t} + \frac{1}{20}\frac{\sin(5\pi t)}{\pi t} \\ X(j\omega) &= \frac{1}{20}[u(\omega+15\pi) - u(\omega-15\pi) + u(\omega+5\pi) - u(\omega-5\pi)] \end{aligned}$$

(d)

$$\begin{aligned} Y(j\omega) &= \frac{1}{2+j7\omega} = \frac{1}{7}\frac{1}{2/7+j\omega} \\ X(j\omega) &= (j\omega)^2 Y(j\omega) \\ x(t) &= \frac{d^2}{dt^2}y(t) = \frac{d^2}{dt^2}\left[\frac{1}{7}e^{-2t/7}u(t)\right] \\ &= \frac{1}{7}\frac{d}{dt}\left[-\frac{2}{7}e^{-2t/7}u(t) + e^{-2t/7}\delta(t)\right] \\ &= \frac{1}{7}\frac{d}{dt}\left[-\frac{2}{7}e^{-2t/7}u(t) + \delta(t)\right] \\ &= \frac{1}{7}\left[\frac{4}{49}e^{-2t/7}u(t) - \frac{2}{7}e^{-2t/7}\delta(t) + \delta^{(1)}(t)\right] \\ &= \frac{1}{7}\left[\frac{4}{49}e^{-2t/7}u(t) - \frac{2}{7}\delta(t) + \delta^{(1)}(t)\right] \end{aligned}$$

(e) Start from the following Fourier transform:

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 3\pi k) = \frac{1}{3}\sum_{k=-\infty}^{+\infty} \frac{2\pi}{2/3}\delta\left(\omega - k\frac{2\pi}{2/3}\right)$$

Its inverse Fourier transform is:

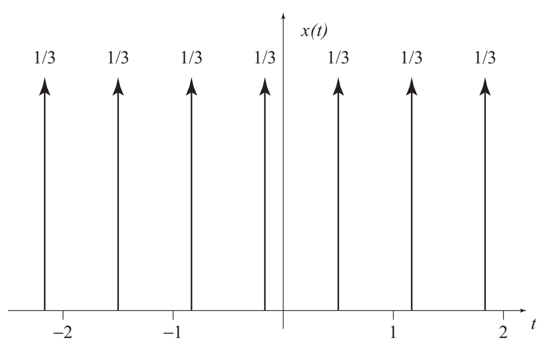
$$p(t) = \frac{1}{3}\sum_{k=-\infty}^{+\infty} \delta(t - 2k/3)$$

(see pp. 368–369 of the textbook). Then:

$$\begin{aligned} \mathcal{F}[p(t+1/6)] &= e^{j\omega/6}P(j\omega) = \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 3\pi k)e^{j\omega/6} \\ &= \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 3\pi k)e^{j3\pi k/6} \\ &= \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 3\pi k)e^{j\pi k/2} \end{aligned}$$

Therefore:

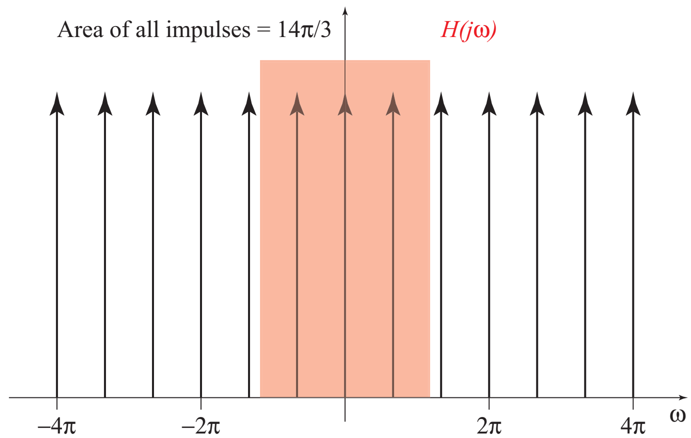
$$x(t) = p(t+1/6) = \frac{1}{3}\sum_{k=-\infty}^{+\infty} \delta(t + 1/6 - 2k/3)$$



Problem 14.2

(a) $X(j\omega) = 7\sum_{n=-\infty}^{+\infty} \frac{2\pi}{3}\delta(\omega - 2n\pi/3)$

(b) $H(j\omega) = \frac{1}{3}e^{-j0.2\omega}[u(\omega + \omega_{co}) - u(\omega - \omega_{co})]$



(c)

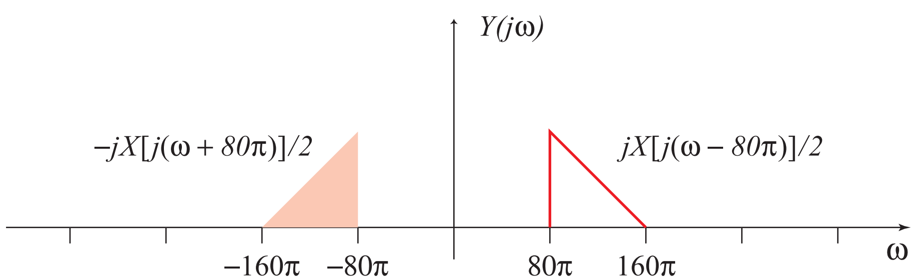
$$
\begin{aligned}
Y(j\omega) &= H(j\omega)X(j\omega) \\
&= \frac{7}{3}\left[\frac{2\pi}{3}\delta(\omega+2\pi/3)e^{-j0.2\omega} + \frac{2\pi}{3}\delta(\omega)e^{-j0.2\omega} + \frac{2\pi}{3}\delta(\omega-2\pi/3)e^{-j0.2\omega}\right] \\
&= \frac{7}{3}\left[\frac{2\pi}{3}\delta(\omega+2\pi/3)e^{j0.4\pi/3} + \frac{2\pi}{3}\delta(\omega) + \frac{2\pi}{3}\delta(\omega-2\pi/3)e^{-j0.4\pi/3}\right] \\
y(t) &= \frac{7}{3}\left(\frac{1}{3}e^{-j2\pi t/3}e^{j0.4\pi/3} + \frac{1}{3} + \frac{1}{3}e^{j2\pi t/3}e^{-j0.4\pi/3}\right) \\
&= \frac{7}{9} + \frac{14}{9}\cos(2\pi t/3 - 0.4\pi/3) = \frac{7}{9} + \frac{14}{9}\cos[2\pi(t-0.2)/3]
\end{aligned}
$$

(d) If $0 < \omega_{co} < 2\pi/3$ then: $y(t) = 7/9$.

Problem 14.3

(a)

$$
\begin{aligned}
v(t) &= x(t)\cos(\omega_m t) = \frac{x(t)}{2}e^{j\omega_m t} + \frac{x(t)}{2}e^{-j\omega_m t} \\
V(j\omega) &= \frac{1}{2}X[j(\omega-\omega_m)] + \frac{1}{2}X[j(\omega+\omega_m)] \\
Y(j\omega) &= H(j\omega)V(j\omega) = \begin{cases} 0 & |\omega| < \omega_m \\ jV(j\omega) & \omega \geq \omega_m \\ -jV(j\omega) & \omega \leq -\omega_m \end{cases}
\end{aligned}
$$

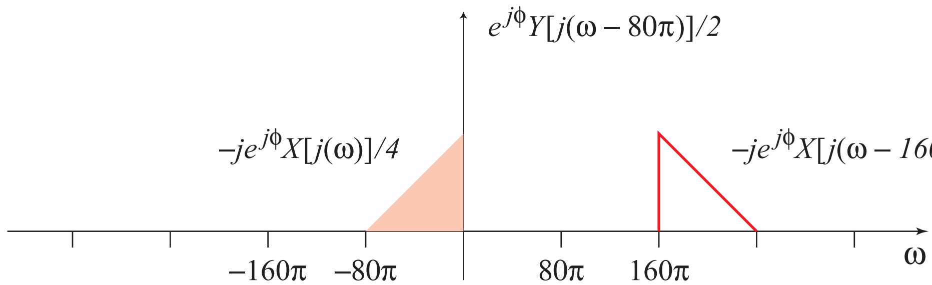


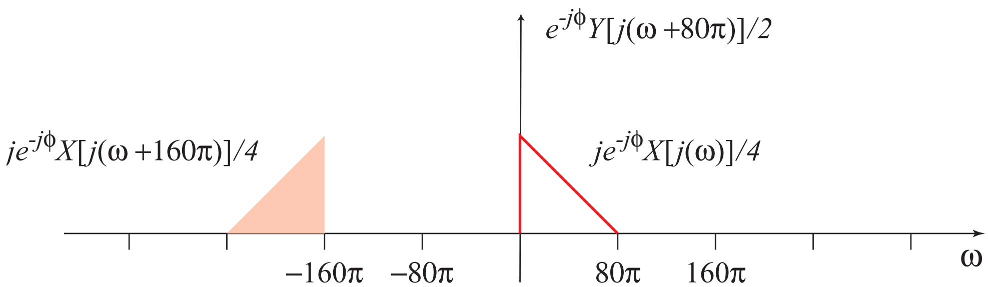
(b) Assume:

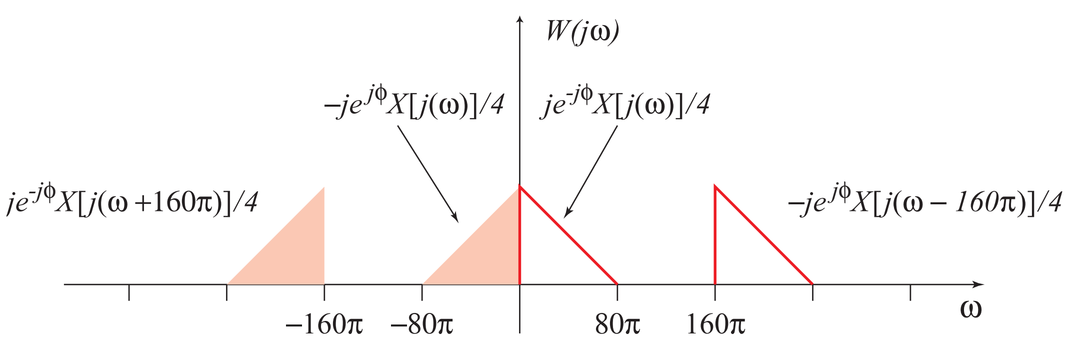
$$w(t) = y(t)\cos(\omega_m t + \phi) = \frac{y(t)}{2}e^{j\phi}e^{j\omega_m t} + \frac{y(t)}{2}e^{-j\phi}e^{-j\omega_m t}$$

Then:

$$W(j\omega) = \frac{1}{2}e^{j\phi}Y[j(\omega - \omega_m)] + \frac{1}{2}e^{-j\phi}Y[j(\omega + \omega_m)]$$







From the picture of $W(j\omega)$ it is apparent that if $\phi = \pi/2$ the central portion of $W(j\omega)$ is equal to $X(j\omega)/4$. Then a low-pass filter with a cutoff frequency between 80π and 160π (with a gain of 4) will recover the original signal $x(t)$.

Problem 14.4

(a)

$$
\begin{aligned}
H(z) &= \frac{z^{-2}}{1+0.8z^{-2}} \\
H(e^{j\hat{\omega}}) &= \frac{e^{-j2\hat{\omega}}}{1+0.8e^{-j2\hat{\omega}}} \\
\hat{\omega} &= \omega/f_s = \omega/200 \\
H_{\text{eff}}(j\omega) &= \frac{e^{-j\omega/100}}{1+0.8e^{-j\omega/100}}
\end{aligned}
$$

This expression for $H_{\text{eff}}(j\omega)$ is valid as long as $|\hat{\omega}| < \pi$, that is, for $|\omega| < \pi f_s = 200\pi$. If the input to the system is: $x(t) = 2\cos(100\pi t)$, then:

$$H_{\text{eff}}(j100\pi) = \frac{e^{-j\pi}}{1+0.8e^{-j\pi}} = -5$$

Therefore: $y(t) = 10\cos(100\pi t + \pi) = -10\cos(100\pi t)$.

(b) The relation $Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$ is satisfied as long as $|\hat{\omega}|_{max} < \pi$. This means $|\omega|_{max}/f_s < \pi$, or: $f_s > |\omega|_{max}/\pi = 80$ samples/sec.

(c) In this case we must have $|\hat{\omega}|_{max} < \pi/2$. This means $|\omega|_{max}/f_s < \pi/2$, or: $f_s > 2|\omega|_{max}/\pi = 160$ samples/sec.

Problem 14.5

(a) We have: $x(t) = x_1(t) + x_2(t) + x_3(t)$, with: $x_1(t) = \cos(22\pi t)$, $x_2(t) = \sin(30\pi t)/2\pi t$, and $x_3(t) = \cos(30\pi t)$. Since $H(j22\pi) = 13e^{-j22\pi/33}$, the output corresponding to $x_1(t)$ is:

$$y_1(t) = 13\cos(22\pi t - 22\pi/33)$$

The Fourier transform of $x_2(t)$ is: $X_2(j\omega) = \frac{1}{2}[u(j\omega + 30\pi) - u(j\omega - 30\pi)]$, and the Fourier transform of the corresponding output is

$$Y_2(j\omega) = H(j\omega)X_2(j\omega) = \frac{1}{2}H(j\omega)$$

because $X_2(j\omega) = 1/2$ for $-30\pi \le \omega \le 30\pi$. Therefore:

$$y_2(t) = \frac{1}{2}h(t) = \frac{13}{2}\frac{\sin[25\pi(t-1/33)]}{\pi(t-1/33)}$$

The output corresponding to $x_3(t)$ is zero because $H(j30\pi) = 0$. Therefore:

$$y(t) = y_1(t) + y_2(t) = 13\cos(22\pi t - 22\pi/33) + \frac{13}{2}\frac{\sin[25\pi(t-1/33)]}{\pi(t-1/33)}$$

(b) As before, $x(t) = x_1(t) + x_2(t)$, with $x_2(t) = 1/2\delta(t)$. Then:

$$y_2(t) = h(t) * x_2(t) = \frac{1}{2}h(t) * \delta(t) = \frac{1}{2}h(t)$$

and:

$$y(t) = y_1(t) + y_2(t) = 13\cos(22\pi t - 22\pi/33) + \frac{13}{2}\frac{\sin[25\pi(t - 1/33)]}{\pi(t - 1/33)}$$

the same as in (a).

(c) Note that $\mathcal{F}[1/2\delta(t)] = 1/2$ is identical to $X_2(j\omega)$ on the range of frequencies where $H(j\omega) \neq 0$.

Problem 14.6