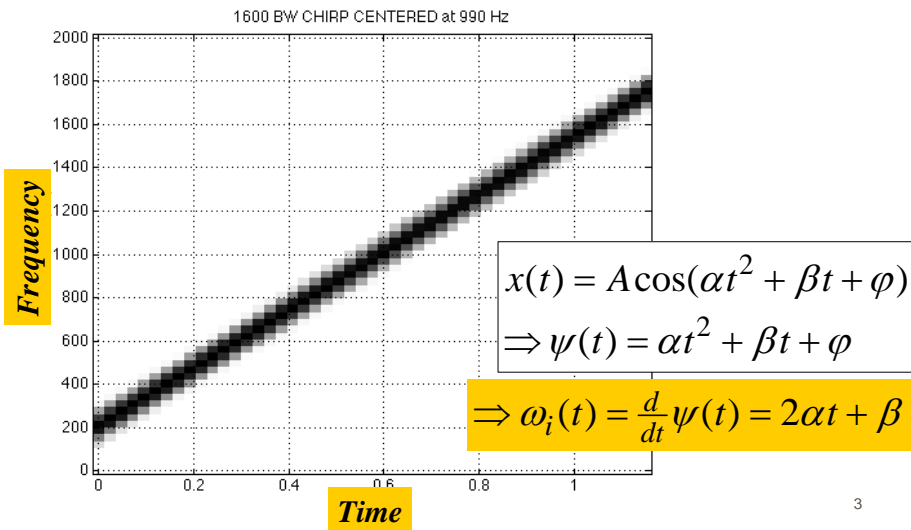


Lecture 6
 Fourier Series Analysis
 30-Jan-2006

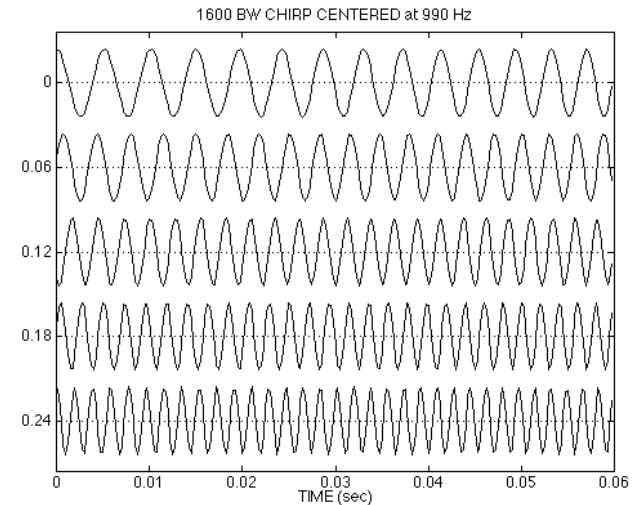
General Info

- **Help Sessions: Sun6, Mon6, Tues6 and Wed6**
 - **Office Hours:** Visit any Prof or TA
 - **Bulletin Board: OFFICIAL ANNOUNCEMENTS**
- **Quiz #1 on 10-Feb**
 - Coverage: HW #1, #2, #3 and #4
 - Old Problems with solutions are on the SP-First CDROM
- Lab #3: **bring headphones**
 - Lots to read about chirps
- Lab #2 report due at beginning of lab
 - **Ask your grading TA about his/her format**
- Prob Set #3 due THIS Week

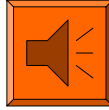
CHIRP SPECTROGRAM 



CHIRP WAVEFORM 

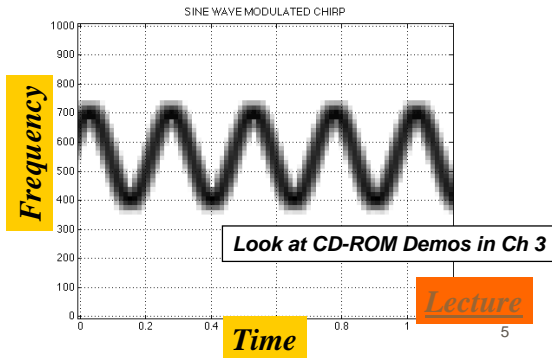


SINE-WAVE FREQUENCY MODULATION (FM)



$$x(t) = A \cos(\alpha \sin(\beta t + \psi) + \varphi)$$

$$\Rightarrow \omega_i(t) = \frac{d}{dt} \psi(t) = \alpha \beta \cos(\beta t + \psi)$$



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READING ASSIGNMENTS

- This Lecture:
 - **Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
 - Replaces pp. 62-66 in Ch 3 in DSP First
 - Notation: \mathbf{a}_k for Fourier Series
- Other Reading:
 - Next Lecture: More Fourier Series

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LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k/T_0)t} dt$$

- **ANALYSIS** via Fourier Series
 - For **PERIODIC** signals: $\mathbf{x}(t+T_0) = \mathbf{x}(t)$
 - Later: spectrum from the Fourier Series

HISTORY

- Jean Baptiste Joseph Fourier
 - 1807 thesis (memoir)
 - On the Propagation of Heat in Solid Bodies
 - Heat !
 - Napoleonic era
- <http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html>

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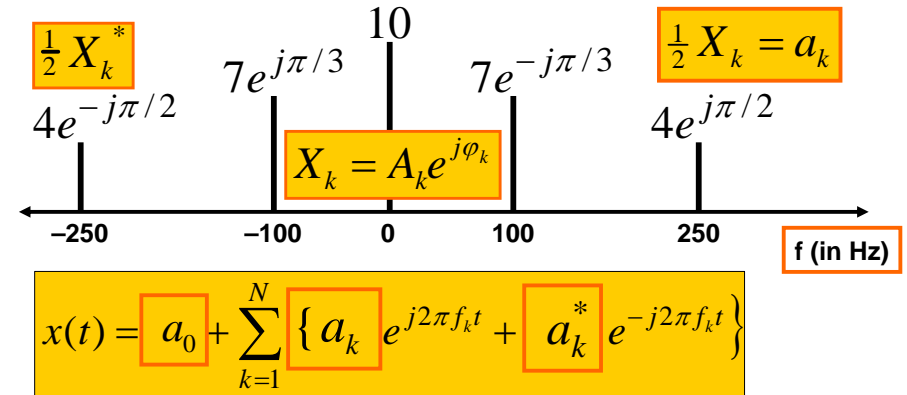
Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

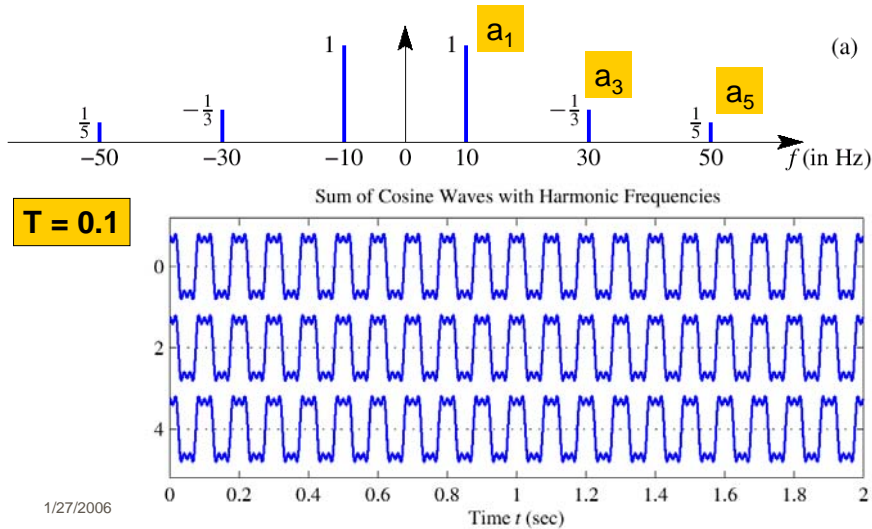
$$a_k = \frac{1}{2} X_k = \frac{1}{2} A_k e^{j\phi_k}$$

$$x(t) = a_0 + \sum_{k=1}^N A_k \cos(2\pi k f_0 t + \phi_k)$$

$$X_k = A_k e^{j\phi_k}$$

COMPLEX AMPLITUDE

Harmonic Signal (3 Freqs)



SYNTHESIS vs. ANALYSIS

- **SYNTHESIS**
 - **Easy**
 - Given (ω_k, A_k, ϕ_k) create $x(t)$
 - Synthesis can be **HARD**
 - Synthesize Speech so that it sounds good
- **ANALYSIS**
 - **Hard**
 - Given $x(t)$, extract (ω_k, A_k, ϕ_k)
 - How many?
 - Need algorithm for computer

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STRATEGY: $x(t) \rightarrow a_k$

- **ANALYSIS**
 - Get representation from the signal
 - Works for **PERIODIC** Signals
- Fourier Series
 - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt = \frac{T_0}{-j2\pi m} e^{-j(2\pi/T_0)mt} \Bigg|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m} (e^{-j2\pi m} - 1)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt = 0$$

$m \neq 0$

$$\omega_0 = \frac{2\pi}{T_0}$$

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ORTHOGONALITY of $\exp(j\omega t)$

- PRODUCT of $\exp(+j\omega t)$ and $\exp(-j\omega t)$

$$\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt} dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t} dt$$

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Isolate One FS Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t} dt = \frac{1}{T_0} \int_0^{T_0} \left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \right) e^{-j(2\pi/T_0)\ell t} dt$$

$$\frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t} dt = \sum_{k=-\infty}^{\infty} a_k \left(\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)(k-\ell)t} dt \right) = a_\ell$$

Integral is zero except for $k = \ell$

$$\Rightarrow a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

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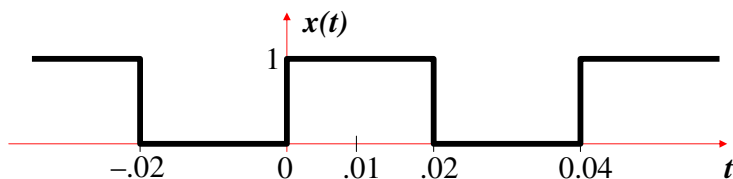
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SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \leq t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



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FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq 0)$$

$$a_k = \frac{1}{.04} \int_0^{.02} 1 e^{-j(2\pi/.04)kt} dt = \frac{1}{.04(-j2\pi k/.04)} e^{-j(2\pi/.04)kt} \Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)} (e^{-j(\pi)k} - 1) = \frac{1 - (-1)^k}{j2\pi k}$$

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DC Coefficient: a_0

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k = 0)$$

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{1}{T_0} (\text{Area})$$

$$a_0 = \frac{1}{.04} \int_0^{.02} 1 dt = \frac{1}{.04} (.02 - 0) = \frac{1}{2}$$

Fourier Coefficients a_k

- a_k is a function of k
 - Complex Amplitude for k -th Harmonic
 - This one doesn't depend on the period, T_0

$$a_k = \frac{1 - (-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi / (0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

