

Lecture 7

Fourier Series & Spectrum

3-Feb-2006

General Info

- **Help Sessions: Sun, M, T, W at 6pm in BH-216**
 - Every week
- **Office Hours:** Visit any Prof or TA
 - See Web-CT for matrix of Office Hours
- **Quiz #1 coming up: Friday, 10-Feb**
- **Bulletin Board: OFFICIAL ANNOUNCEMENTS**
 - Prob Set #4 due next Week; solution on Thursday
 - Lab #3 due starting on Tuesday

Sinusoidal Synthesis

- Use Short-Duration Sinusoids:
 - Amp, Phase, Frequency & Duration

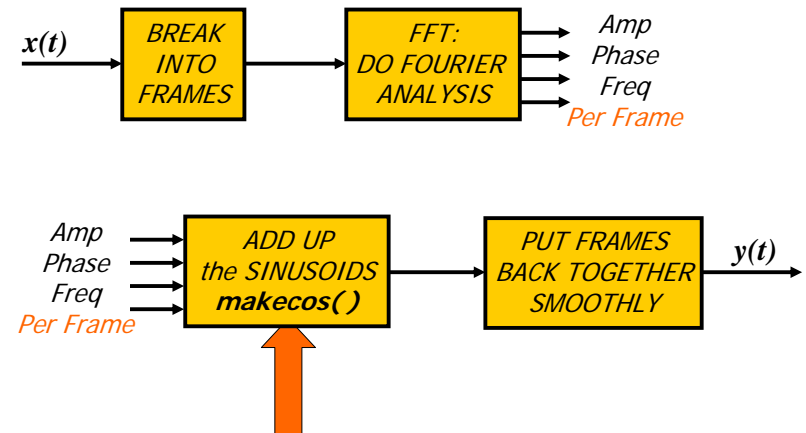
$$x(t) = A_k \cos(2\pi f_k t + \phi_k) \text{ for } t_k \leq t \leq t_{k+1}$$

- Freq will change every FRAME





$$t_k \leq t \leq t_{k+1}$$

- Then ADD several sinusoids together

ANALYSIS --> SYNTHESIS



Sine Synthesis: SPEECH

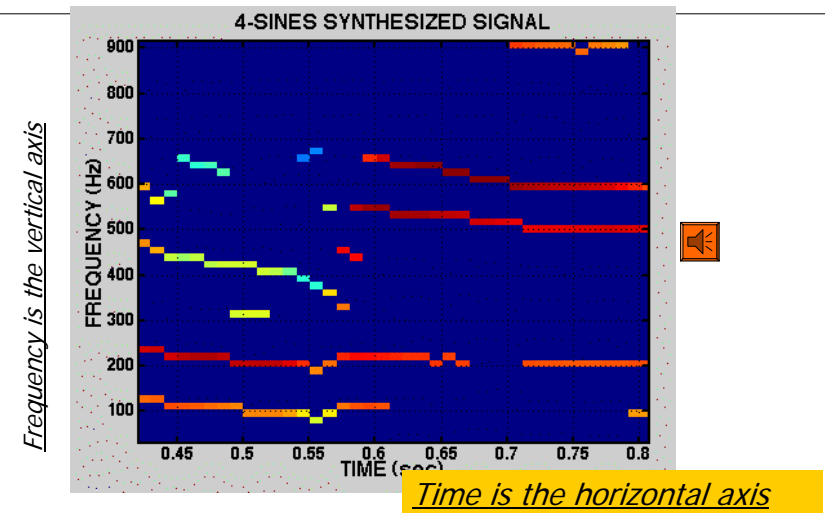
- FRAME Length = 10 millisec
- Examples:
 - Original 
 - 9 sinusoids per frame 
 - 4 sinusoids 
 - 2 sinusoids 
- Need to **SMOOTH** Boundaries

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Time-Varying FREQUENCIES "Diagram"

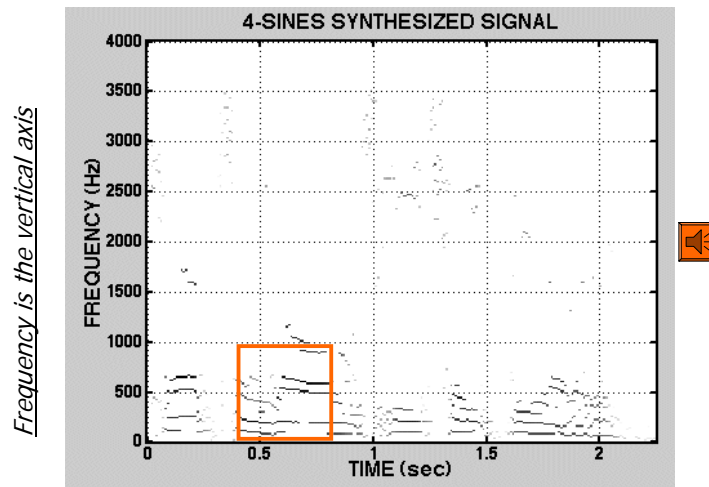


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4-SINES Spectrogram

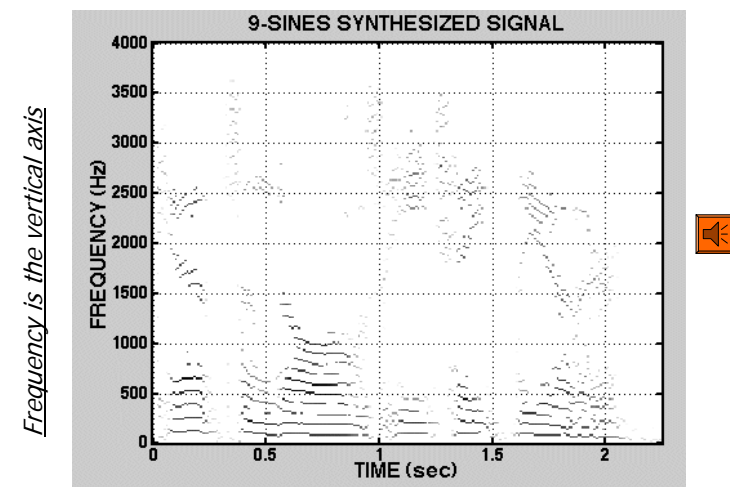


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9-SINES Spectrogram

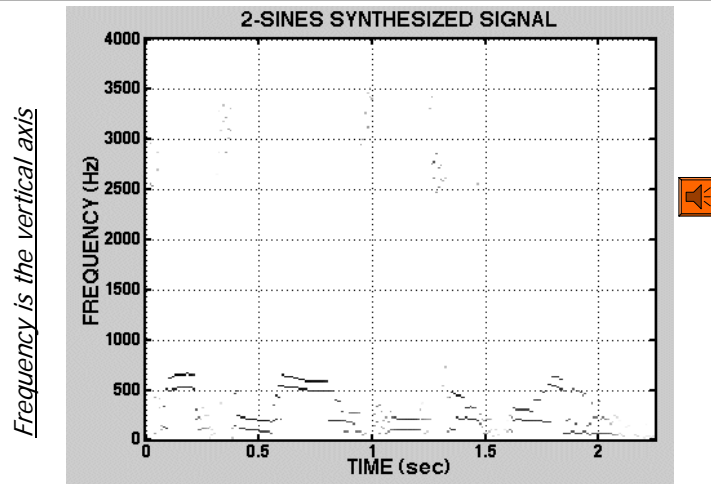


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2-SINES Spectrogram

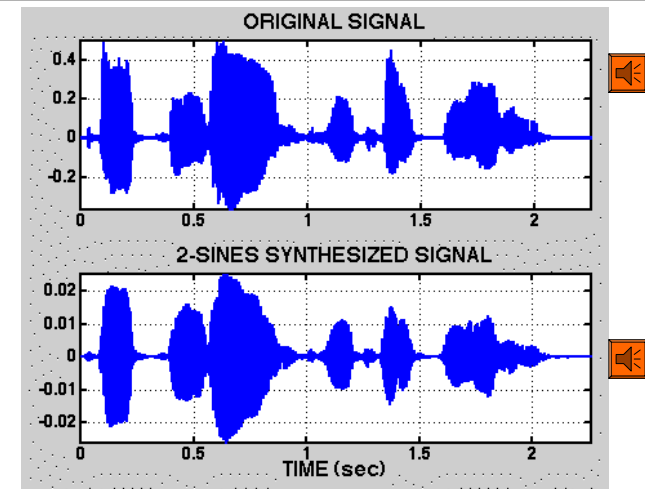


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TIME SIGNALS: COMPARE

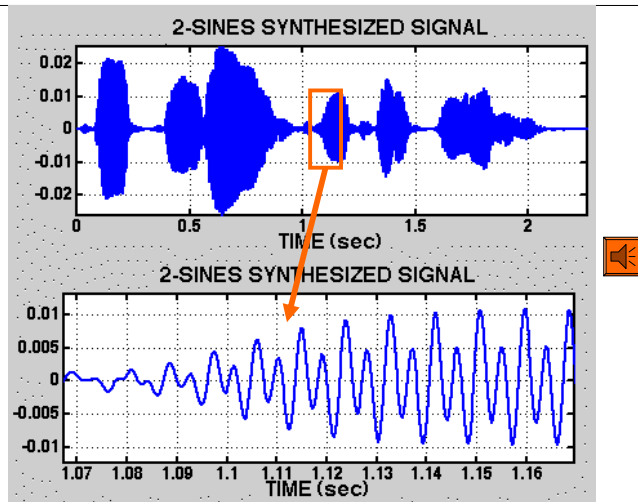


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TIME SIGNALS: ZOOM



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READING ASSIGNMENTS

- This Lecture:
 - **Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
 - Replaces pp. 62-66 in Ch 3 in DSP First
 - Notation: \mathbf{a}_k for Fourier Series
- Other Reading:
 - Next Lecture: Sampling

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LECTURE OBJECTIVES

- **ANALYSIS** via Fourier Series

- For **PERIODIC** signals: $x(t+T_0) = x(t)$

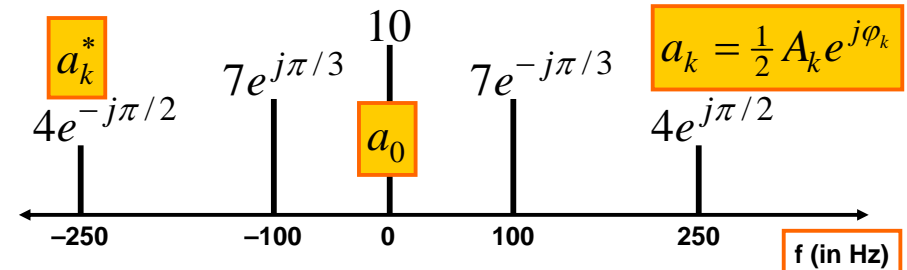
$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k/T_0)t} dt$$

- **SPECTRUM** from Fourier Series

- a_k is Complex Amplitude for k-th Harmonic

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^N \left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right\}$$

Harmonic Signal

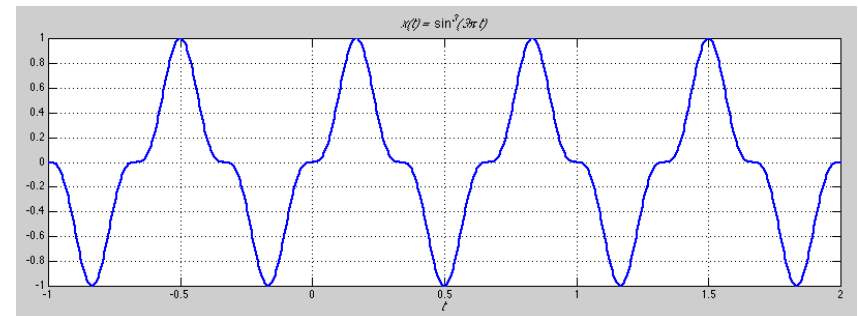
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Example

$$x(t) = \sin^3(3\pi t)$$



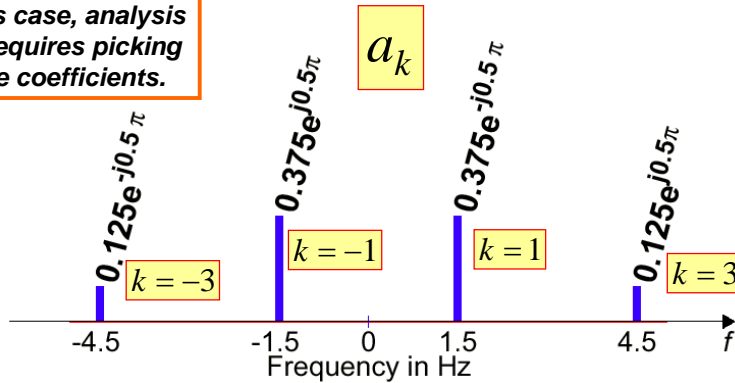
$$x(t) = \left(\frac{j}{8}\right) e^{j9\pi t} + \left(\frac{-3j}{8}\right) e^{j3\pi t} + \left(\frac{3j}{8}\right) e^{-j3\pi t} + \left(\frac{-j}{8}\right) e^{-j9\pi t}$$

Example

$$x(t) = \sin^3(3\pi t)$$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

In this case, analysis just requires picking off the coefficients.



STRATEGY: $x(t) \rightarrow a_k$

ANALYSIS

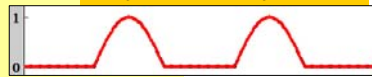
- Get representation from the signal
- Works for **PERIODIC** Signals
- Fourier Series
 - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq \pm 1)$$

Half-Wave Rectified Sine



$$\begin{aligned} a_k &= \frac{1}{T_0} \int_0^{T_0/2} \sin\left(\frac{2\pi}{T_0}t\right) e^{-j(2\pi/T_0)kt} dt \\ &= \frac{1}{T_0} \int_0^{T_0/2} \frac{e^{j(2\pi/T_0)t} - e^{-j(2\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt} dt \\ &= \frac{1}{j2T_0} \int_0^{T_0/2} e^{-j(2\pi/T_0)(k-1)t} dt - \frac{1}{j2T_0} \int_0^{T_0/2} e^{-j(2\pi/T_0)(k+1)t} dt \\ &= \frac{e^{-j(2\pi/T_0)(k-1)T_0/2}}{j2T_0(-j(2\pi/T_0)(k-1))} - \frac{e^{-j(2\pi/T_0)(k+1)T_0/2}}{j2T_0(-j(2\pi/T_0)(k+1))} \end{aligned}$$

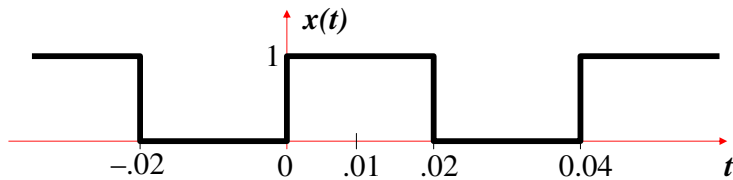
FS: Rectified Sine Wave $\{a_k\}$

$$\begin{aligned} a_k &= \frac{e^{-j(2\pi/T_0)(k-1)T_0/2}}{j2T_0(-j(2\pi/T_0)(k-1))} - \frac{e^{-j(2\pi/T_0)(k+1)T_0/2}}{j2T_0(-j(2\pi/T_0)(k+1))} \\ &= \frac{1}{4\pi(k-1)} \left(e^{-j(2\pi/T_0)(k-1)T_0/2} - 1 \right) - \frac{1}{4\pi(k+1)} \left(e^{-j(2\pi/T_0)(k+1)T_0/2} - 1 \right) \\ &= \frac{1}{4\pi(k-1)} \left(e^{-j\pi(k-1)} - 1 \right) - \frac{1}{4\pi(k+1)} \left(e^{-j\pi(k+1)} - 1 \right) \\ &= \begin{cases} 0 & k \text{ odd} \\ ? & k = \pm 1 \\ \frac{-1}{2\pi(k^2-1)} & k \text{ even} \end{cases} \end{aligned}$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \leq t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



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Fourier Coefficients a_k

- a_k is a function of k
 - Complex Amplitude for k -th Harmonic
 - This one doesn't depend on the period, T_0

$$a_k = \frac{1 - (-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

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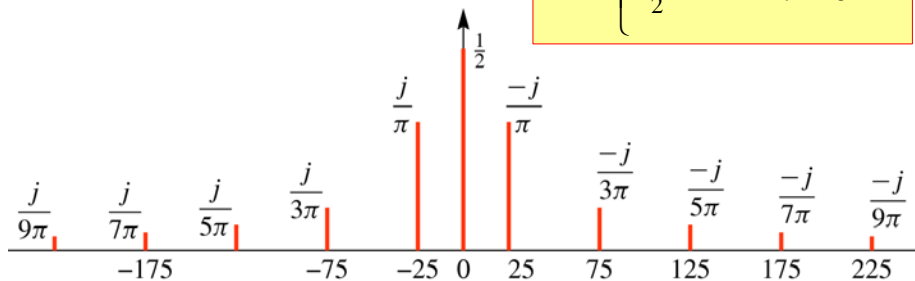
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Spectrum from Fourier Series

$$\omega_0 = 2\pi / (0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$



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Fourier Series Synthesis

- HOW do you **APPROXIMATE** $x(t)$?

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

- Use FINITE number of coefficients

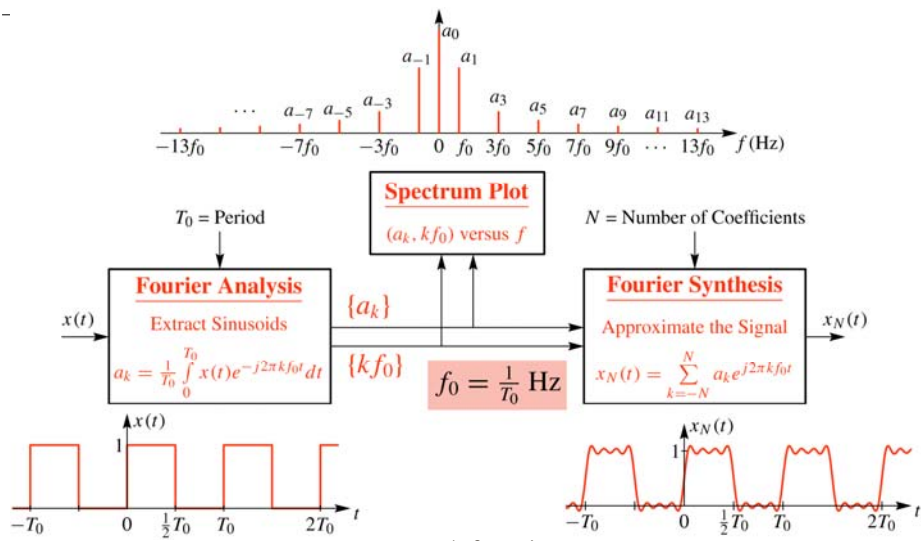
$$x(t) = \sum_{k=-N}^N a_k e^{j2\pi k f_0 t} \quad a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}$$

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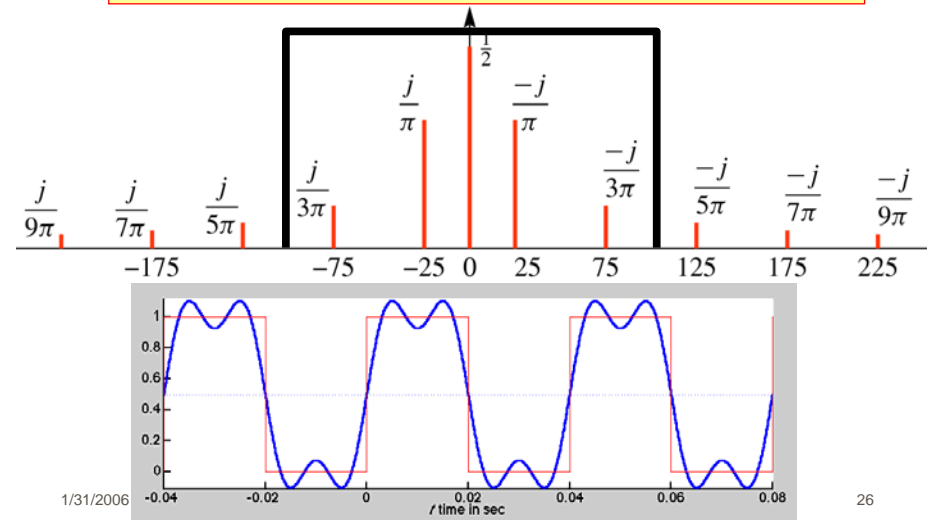
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Fourier Series Synthesis



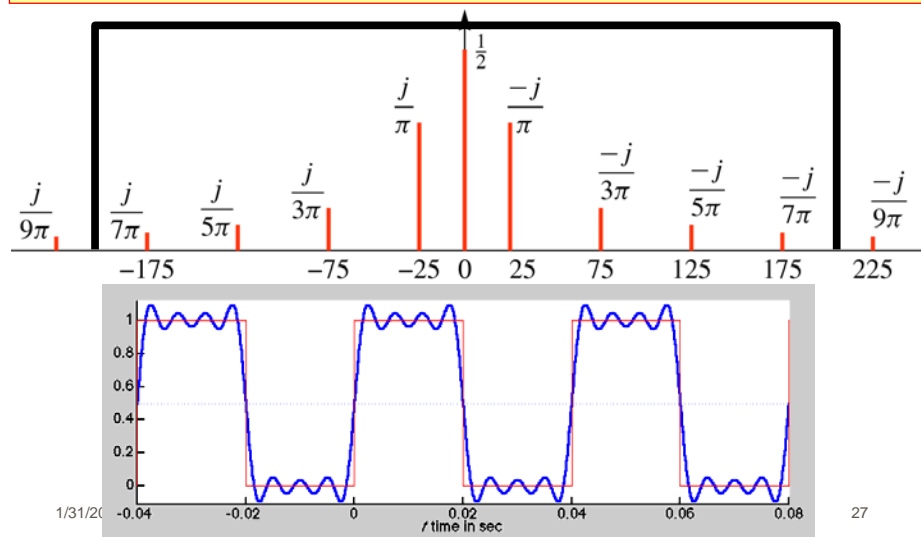
Synthesis: 1st & 3rd Harmonics

$$y(t) = \frac{1}{2} + \frac{2}{\pi} \cos(2\pi(25)t - \frac{\pi}{2}) + \frac{2}{3\pi} \cos(2\pi(75)t - \frac{\pi}{2})$$



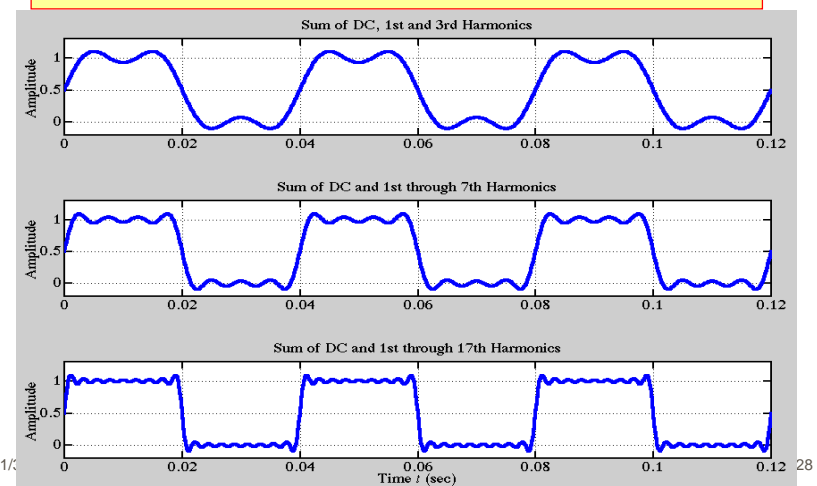
Synthesis: up to 7th Harmonic

$$y(t) = \frac{1}{2} + \frac{2}{\pi} \cos(50\pi t - \frac{\pi}{2}) + \frac{2}{3\pi} \sin(150\pi t) + \frac{2}{5\pi} \sin(250\pi t) + \frac{2}{7\pi} \sin(350\pi t)$$



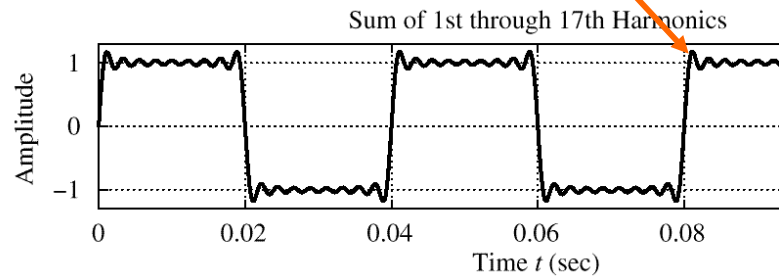
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi} \sin(\omega_0 t) + \frac{2}{3\pi} \sin(3\omega_0 t) + \dots$$



Gibbs' Phenomenon

- Convergence at **DISCONTINUITY** of $x(t)$
 - There is always an **overshoot**
 - 9%** for the Square Wave case



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Fourier Series Demos

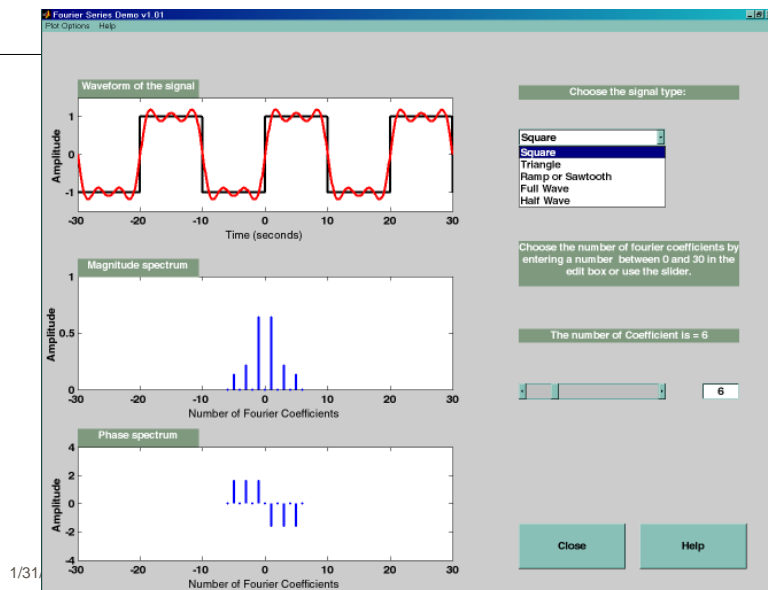
- Fourier Series Java Applet
 - Greg Slabaugh
 - Interactive
 - <http://users.ece.gatech.edu/~slabaugh/java/fourier/fourier.html>
- MATLAB GUI: fseriesdemo
 - <http://users.ece.gatech.edu/mcclella/matlabGUIs/index.html>

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fseriesdemo GUI



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