ECE 2025, Spring 2006, Problem Set #6

- Problem 6.1

A linear time-invariant discrete-time system is described by the difference equation

$$y[n] = -x[n] + x[n-2] - 3x[n-4] + 3x[n-5]$$

(a) To find the impulse response $h[n]$, we simply replace $x[n]$ by $\delta[n]$, and $y[n]$ by $h[n]$:

$$h[n] = -\delta[n] + \delta[n-2] - 3\delta[n-4] + 3\delta[n-5]$$

(b) If we write $y[n] = \sum_{k=0}^{M} b_k x[n-k]$, then

$$b_k = \begin{cases} -1, & k = 0, \\ 1, & k = 2, \\ -3, & k = 4, \\ 3, & k = 5, \\ 0, & \text{otherwise.} \end{cases}$$

(c) The order of the filter $M = 5$, the length of the filter $L = M+1 = 6$.

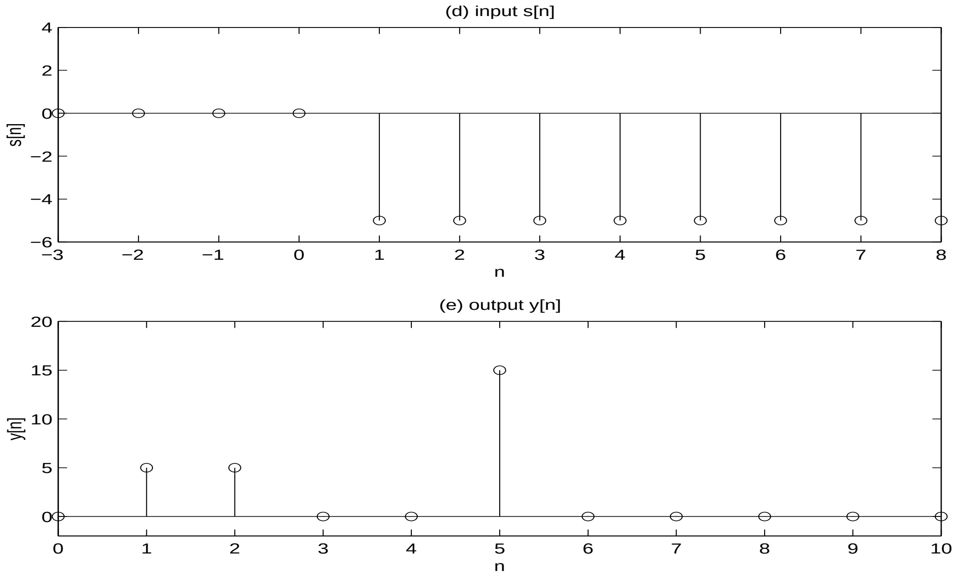
(d)

$$s[n] = -5u[n-1] = \begin{cases} -5, & n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Plot of $s[n]$: see subplot (d) on next page.

(e) Use the convolution table to determine the output due to the input $s[n] = -5u[n-1]$.

n	0	1	2	3	4	5	6	7	8	9	10
x[n]		-5	-5	-5	-5	-5	-5	-5	-5	-5	-5
h[n]	-1	0	1	0	-3	3					
h[0]x[n-0]		5	5	5	5	5	5	5	5	5	5
h[2]x[n-2]				-5	-5	-5	-5	-5	-5	-5	-5
h[4]x[n-4]						15	15	15	15	15	15
h[5]x[n-5]							-15	-15	-15	-15	-15
y[n]		5	5	0	0	15	0	0	0	0	0



Although the above convolution table extends to $n \to \infty$, we can infer from the pattern above that $y[n] = 0$ for all $n \geq 6$.

Plot the output sequence $y[n]$ for $0 \leq n \leq 10$: see subplot (e).

- Problem 6.2

System #1:

$$y[n] = \pi x[n-1]$$

System #2:

$$y[n] = 2^{-x[n+3]}$$

(a) To find the impulse response, we substitute $x[n]$ by $\delta[n]$ and $y[n]$ by $h[n]$:

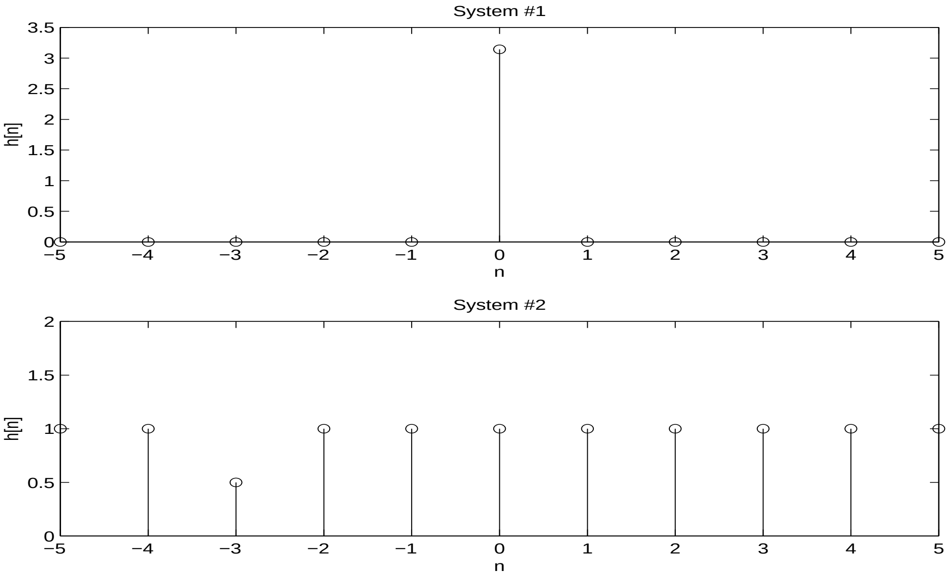
System #1:

$$h[n] = \pi\delta[n-1] = \begin{cases} \pi, & n = 1, \\ 0, & \text{otherwise} \end{cases}$$

System #2:

$$h[n] = 2^{-\delta[n+3]} = \begin{cases} 0.5, & n = -3, \\ 1, & \text{otherwise} \end{cases}$$

The impulse responses are shown in the plots below.



(b) Linearity check for System #1:

If $x_1[n]$ is input to the system, the corresponding output is $y_1[n] = \pi x_1[n-1]$.

If $x_2[n]$ is input to the system, the corresponding output is $y_2[n] = \pi x_2[n-1]$.

If $ax_1[n] + bx_2[n]$ is input to the system, the corresponding output is $z[n] = \pi(ax_1[n-1] + bx_2[n-1])$.

Since $z[n] = ay_1[n] + by_2[n]$, System #1 is linear.

Linearity check for System #2:

If $x_1[n]$ is input to the system, the corresponding output is $y_1[n] = 2^{-x_1[n+3]}$.

If $x_2[n]$ is input to the system, the corresponding output is $y_2[n] = 2^{-x_2[n+3]}$.

If $ax_1[n] + bx_2[n]$ is input to the system, the corresponding output is $z[n] = 2^{-(ax_1[n+3]+bx_2[n+3])}$

Since $z[n] \neq ay_1[n] + by_2[n]$, System #2 is not linear.

(c) Time-invariance check for System #1:

If $x[n]$ is input to the system, the corresponding output is $y[n] = \pi x[n-1]$.

If $x_1[n] = x[n-n_d]$ is input to the system, the corresponding output is $y_1[n] = \pi x_1[n-1] = \pi x[n-n_d-1]$.

Since $y_1[n] = y[n-n_d]$, System #1 is time-invariant.

Time-invariance check for System #2:

If $x[n]$ is input to the system, the corresponding output is $y[n] = 2^{-x[n+3]}$.

If $x_1[n] = x[n-n_d]$ is input to the system, the corresponding output is $y_1[n] = 2^{-x_1[n+3]} = 2^{-x[n-n_d+3]}$.

Since $y_1[n] = y[n-n_d]$, System #2 is time-invariant.

(d) Causality check for System #1:

If $x[n] = 0$ for $n \leq n_0$, then $y[n] = \pi x[n-1] = 0$ for all $n \leq n_0 + 1$, which includes the $n \leq n_0$ region. Thus System #1 is causal.

Causality check for System #2:

(Counter-example): If $x[n] = \delta[n]$, then $x[n] = 0$ for $n < 0$, but $y[n] \neq 0$, $\forall n$ (see the impulse response plot shown earlier). Thus, System #2 is non-causal.

- Problem 6.3

Impulse response for LTI System #1:

$$h_1[n] = 5u[n] - 5u[n-3]$$

LTI System #2 is described by the difference equation

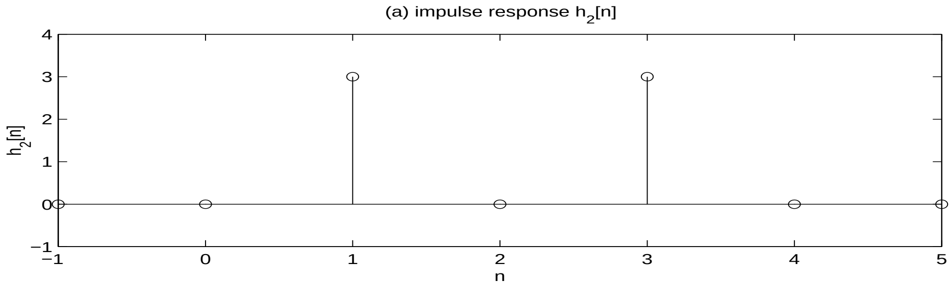
$$y_2[n] = 3y_1[n-3] + 3y_1[n-1]$$

(a) To find the impulse response sequence, $h_2[n]$, of the second system, we simply replace $y_1[n]$ by $\delta[n]$ and $y_2[n]$ by $h_2[n]$:

$$h_2[n] = 3\delta[n-3] + 3\delta[n-1] = 3\delta[n-1] + 3\delta[n-3]$$

Plot $h_2[n]$ versus n: see subplot (a) below.

(b) Determine the impulse response of the overall cascade system $h[n] = h_1[n] * h_2[n]$.



First, we rewrite $h_1[n]$ as

$$h_1[n] = 5u[n] - 5u[n-3] = 5\delta[n] + 5\delta[n-1] + 5\delta[n-2]$$

$$\begin{aligned} & h[n] = h_1[n] * h_2[n] \\ = \; & (5\delta[n] + 5\delta[n-1] + 5\delta[n-2]) * (3\delta[n-1] + 3\delta[n-3]) \\ = \; & (15\delta[n-1] + 15\delta[n-2] + 15\delta[n-3]) + (15\delta[n-3] + 15\delta[n-4] + 15\delta[n-5]) \\ = \; & 15\delta[n-1] + 15\delta[n-2] + 30\delta[n-3] + 15\delta[n-4] + 15\delta[n-5] \end{aligned}$$

(c) If the input signal is $x[n] = \cos(0.5\pi n - 0.25\pi)(u[n] - u[n-12])$, determine the output signal $y[n]$.

First, the $(u[n] - u[n-12])$ part limits the non-zero region of $x[n]$ to $0 \le n \le 11$.

The $\cos(0.5\pi n - 0.25\pi)$ part is periodic in n with period $1/\hat{f}_0 = 1/(0.25) = 4$. In fact, $x[n]$ equals $\frac{\sqrt{2}}{2}$ $[1 \;\; 1 \;\; -1 \;\; -1]$ for $0 \le n \le 3$ and then repeats 2 more times during $4 \le n \le 11$.

The output $y[n] = x[n] * h_1[n] * h_2[n]$. Since convolution is commutative, we can write

$$\begin{aligned} y[n] &= \frac{\sqrt{2}}{2} \underbrace{[1 \;\; 1 \;\; -1 \;\; -1 \;\; 1 \;\; 1 \;\; -1 \;\; -1 \;\; 1 \;\; 1 \;\; -1 \;\; -1]}_{x_1[n]} * h_2[n] * h_1[n] \\ &= \frac{\sqrt{2}}{2} \underbrace{(x_1[n] * h_2[n])}_{z[n]} * h_1[n] \\ &= \frac{\sqrt{2}}{2} z[n] * h_1[n] \end{aligned}$$

First, let us use the convolution table to find $z[n] = x_1[n] * h_2[n]$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x1[n]	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1			
h2[n]		3	0	3											
h2[1]x1[n-1]		3	3	-3	-3	3	3	-3	-3	3	3	-3	-3		
h2[3]x1[n-3]				3	3	-3	-3	3	3	-3	-3	3	3	-3	-3
z[n]		3	3	0	0	0	0	0	0	0	0	0	0	-3	-3

Therefore, $z[n] = 3\delta[n-1] + 3\delta[n-2] - 3\delta[n-13] - 3\delta[n-14]$.

Finally,

$$
\begin{aligned}
y[n] &= \frac{\sqrt{2}}{2}\,(z[n] * h_1[n]) \\
&= \frac{15\sqrt{2}}{2}\,(\delta[n-1] + \delta[n-2] - \delta[n-13] - \delta[n-14]) * (\delta[n] + \delta[n-1] + \delta[n-2]) \\
&= \frac{15\sqrt{2}}{2}\,(\delta[n-1] + \delta[n-2] - \delta[n-13] - \delta[n-14]) \\
&\quad + \frac{15\sqrt{2}}{2}\,(\delta[n-2] + \delta[n-3] - \delta[n-14] - \delta[n-15]) \\
&\quad + \frac{15\sqrt{2}}{2}\,(\delta[n-3] + \delta[n-4] - \delta[n-15] - \delta[n-16]) \\
&= 10.6066\,(\delta[n-1] + 2\delta[n-2] + 2\delta[n-3]) + \delta[n-4]) \\
&\quad -10.6066\,(\delta[n-13] + 2\delta[n-14] + 2\delta[n-15] + \delta[n-16])
\end{aligned}
$$

Thus, the non-zero regions of $y[n]$ are $1 \leq n \leq 4$ and $13 \leq n \leq 16$; $y[n]$ is zero elsewhere.

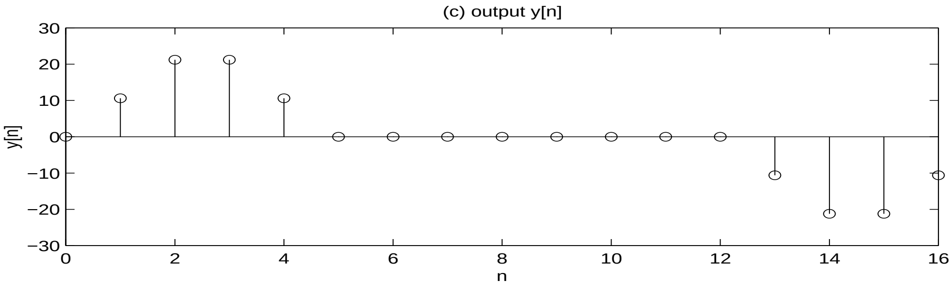
Plot of $y[n]$ versus n: see subplot (c) below.

- Problem 6.4

  The input $x[n]$ is zero for $n < 0$.

  The system is defined by

$$
y[n] = \sum_{k=n-100}^{n-1} \frac{x[k]}{n-k}
$$



(a) To find the impulse response, we replace $x[n]$ by $\delta[n]$ and $y[n]$ by $h[n]$:

$$h[n] = \sum_{k=n-100}^{n-1} \frac{\delta[k]}{n-k}$$

Since $\delta[k] \neq 0$ only when $k = 0$, if

$$n - 100 \leq (k = 0) \leq n - 1$$

i.e., if $1 \leq n \leq 100$, then

$$h[n] = \left. \frac{\delta[k]}{n-k} \right|_{k=0} = \frac{1}{n}.$$

If $k = 0$ is not included in the interval $[n-100, n-1]$; i.e., if $n < 1$ or $n > 100$, then $h[n] = 0$.

In summary,

$$h[n] = \frac{1}{n}(u[n-1] - u[n-101]) = \begin{cases} \frac{1}{n}, & 1 \leq n \leq 100, \\ 0, & \text{otherwise.} \end{cases}$$

Verification:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k] \frac{1}{n-k} \left(u[n-k-1] - u[n-k-101]\right) \end{aligned}$$

The term $u[n-k-1] - u[n-k-101]$ constrains $1 \le n-k \le 100$ $\Rightarrow\ n-100 \le k \le n-1$. Thus,

$$y[n] = \sum_{k=n-100}^{n-1} \frac{x[k]}{n-k}$$

(b) The filter length is $L_h = 100$. The filter coefficients

$$b_k = h[k] = \begin{cases} \frac{1}{k}, & 1 \le k \le 100, \\ 0, & \text{otherwise} \end{cases}$$

(c) The input $x[n]$ has positive values for $50 \le n \le 300$, but is zero elsewhere. Determine the interval where the output $y[n]$ will be nonzero.

From

$$y[n] = \sum_{k=n-100}^{n-1} \frac{x[k]}{n-k}$$

we infer that if the end of the summation window, $k = n-1$, is to the left of $k = 50$, then $y[n] = 0$; i.e., if $n-1 < 50$, or, $n < 51$, then $y[n] = 0$.

On the other hand, if the beginning of the summation window, $k = n-100$, is to the right of $k = 300$, then $y[n] = 0$; i.e., if $n-100 > 300$, or, $n > 400$, then $y[n] = 0$.

In summary, $y[n] \ne 0$ for $51 \le n \le 400$.

(d) The length of the input signal $L_x = 300 - 50 + 1 = 251$.

The length of the impulse response $L_h = 100 - 1 + 1 = 100$.

The length of the output signal is $L_y = 400 - 51 + 1 = 350$.

The above lengths satisfy $L_y = L_x + L_h - 1$.

- Problem 6.5

(a) Recall that $x[n] * \delta[n-n_0] = x[n-n_0]$. Therefore,

$$y[n] = \delta[n-31]*\delta[n-41]*\delta[n-59] = \delta[n-72]*\delta[n-59] = \delta[n-131]$$

(b) Use the convolution table to determine $y[n] = u[n] * u[n]$ and give a general formula.

n	0	1	2	3	4
x[n]=u[n]	1	1	1	1	1
h[n]=u[n]	1	1	1	1	1
h[0]x[n-0]	1	1	1	1	1
h[1]x[n-1]		1	1	1	1
h[2]x[n-2]			1	1	1
h[3]x[n-3]				1	1
h[4]x[n-4]					1
y[n]	1	2	3	4	5

By inference, we find

$$y[n] = u[n] * u[n] = (n+1)u[n].$$

(c) Generalize the previous part to get a simple formula for

$$y[n] = u[n] * u[n] * u[n]$$

Incorporating the result from (b), we find

$$y[n] = ((n+1)u[n]) * u[n]$$

Let $x[n] = (n+1)u[n]$, $h[n] = u[n]$, and write the convolution table,

n	0	1	2	3	4
x[n]	1	2	3	4	5
h[n]	1	1	1	1	1
h[0]x[n-0]	1	2	3	4	5
h[1]x[n-1]		1	2	3	4
h[2]x[n-2]			1	2	3
h[3]x[n-3]				1	2
h[4]x[n-4]					1
y[n]	1	3	6	10	15

By inference, we find

$$y[n] = ((n+1)u[n]) * u[n] = \left(\sum_{k=1}^{n+1} k\right) u[n] = \frac{(n+1)(n+2)}{2}\, u[n].$$