ECE 2025, Georgia Tech

Solution to PS # 9

9.1

a) $x_a[n] = 5\delta[n-1] + 3^n u[n-2]$

$$x_a[n] = 5\delta[n-1] + 3^2 (3)^{n-2} u[n-2]$$

$$X_a(z) = 5z^{-1} + 9 \frac{z^{-2}}{1-3z^{-1}} = \frac{5z^{-1} - 6z^{-2}}{1-3z^{-1}}$$

b) $x_b[n] = 2(-\frac{1}{2})^n u[n] + 3(\frac{-1}{3})^n u[n]$

$$X_b(z) = 2\frac{1}{1+\frac{1}{2}z^{-1}} + 3\frac{1}{1+\frac{1}{3}z^{-1}} = \frac{5 + \frac{13}{6}z^{-1}}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}$$

c) $x_c[n] = n\, u[n]$

$$u[n] * u[n] = \sum_{k=0}^{\infty} u[k]\, u[n-k] = \sum_{k=0}^{n} 1 = n+1 \quad \text{for } n \geq 0$$

Thus $u[n] * u[n] = (n+1)\, u[n] \rightarrow x_c[n] = u[n] * u[n] - u[n]$

$$\rightarrow X_c(z) = \frac{1}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} = \frac{z^{-1}}{(1-z^{-1})^2}$$

9.2

a) $Y(z) = -0.9 z^{-1} Y(z) + X(z) - z^{-1} X(z)$

$\rightarrow Y(z)(1+0.9z^{-1}) = X(z)(1-z^{-1})$

$\rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1+0.9z^{-1}} = \frac{1}{1+0.9z^{-1}} - \frac{z^{-1}}{1+0.9z^{-1}}$

$\rightarrow h[n] = (-0.9)^{n} u[n] - (-0.9)^{n-1} u[n-1]$

b) $x[n] = u[n-2] \longrightarrow X(z) = \frac{z^{-2}}{1-z^{-1}}$

$Y(z) = H(z) X(z) \longrightarrow Y(z) = \frac{1-z^{-1}}{1+0.9z^{-1}} \cdot \frac{z^{-2}}{1-z^{-1}} = \frac{z^{-2}}{1+0.9z^{-1}}$

$\rightarrow y[n] = (-0.9)^{n-2} u[n-2]$

c)

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bb= [1  -1] ;
aa= [1  0.9] ;
xx= [0  0  ones(1,9)] ;
yy = filter(bb, aa, xx) ;          index = 0:1:(length(yy)-1);
stem (yy, index)
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9.3

a) $H(e^{j\hat{\omega}}) = \dfrac{1-(e^{j\hat{\omega}})^{-1}}{1+0.9(e^{j\hat{\omega}})^{-1}} = \dfrac{1-e^{-j\hat{\omega}}}{1+.9e^{-j\hat{\omega}}}$

b) You may use matlab to plot this. However we can also approximately polt it using the pole-zero locations.

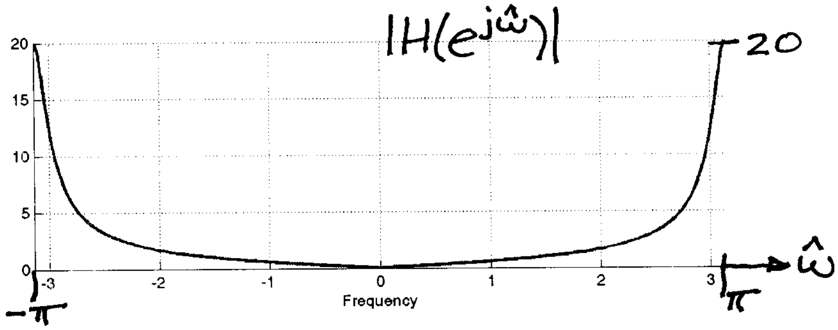
Note that $|H(e^{j\hat{\omega}})|^2 = \dfrac{1-e^{-j\hat{\omega}}}{1+0.9\,e^{-j\hat{\omega}}} \cdot \dfrac{1-e^{j\hat{\omega}}}{1+0.9\,e^{j\hat{\omega}}} = \dfrac{2(1-\cos\hat{\omega})}{1.81+1.8\cos\hat{\omega}}$

$H(z)$ has one zero at $z=1$ and a pole at $z=-0.9$

Thus, $|H(e^{j\hat{\omega}})| = 0$ at $\hat{\omega}=0$ and $|H(e^{j\hat{\omega}})|$ is large at $\hat{\omega}=\pi$
(i.e., has a peak)

That is $|H(e^{j\pi})|^2 = \dfrac{2(2)}{1.81-1.8} = 400$

The plot is even symmetric w.r.t $\hat{\omega}$.



c) $H(e^{j0.8\pi}) = \dfrac{1-e^{-j0.8\pi}}{1+0.9\,e^{-j0.8\pi}} \simeq 3.2\,e^{j0.45\pi}$

$\Rightarrow y[n] = 320\cos(0.8\pi n - 0.2\pi + 0.45\pi)$
$= 320\cos(0.8\pi n + 0.25\pi)$

9.4

Freq. Response A : should have a zero at $\hat{\omega}=0$, i.e $z=1$.
Thus, S_3, S_4, S_6, S_8 are possible candidates.
However, $|H(e^{j\omega})|$ at $\hat{\omega}=\pi$ for S_3, S_6, S_8 are
$8, 4, \frac{8}{3}$, respectively. Hence, the correct answer is S_4
for which $|H(e^{j\hat{\omega}})| = 1.2$ at $\hat{\omega}=\pi$

Freq. Res. B : Must have zero at $Z=-1$ (i.e., $\hat{\omega}=\pi$).

possible answers are S_1, S_2, S_7. Hower, only S_2
satisfies the condition $|H(e^{j0})| = H(z)\big|_{z=1} = 8$

Freq. Res. C : should have a complex conjugate zero (at $0<\hat{\omega}<\frac{\pi}{2}$). Thus, possible candidate is S_5.

All the other systems have either zero at $\omega=0$ or $\hat{\omega}=\pi$.

Freq. Res. D : Must have zero at $z=-1$ ($\hat{\omega}=\pi$).

possible answer : S_1, S_2, S_7.

However, only S_1 satisfies the magnitude condition
at $\hat{\omega}=0$ $|H(e^{j0})| = \frac{20}{6}$

Freq. Res. E : Must have a zero at $z=1$ and a zero at
$z=j$ ($\hat{\omega}=\frac{\pi}{2}$). Only S_6 satisfies this.

Freq. Res. F : zero at $z=-1$ and $H(1)=0.5$ only
S_7 satisfies these conditions. (we also expect a zero close to
unit circle at $0<a<1$.

9.5

a) $H_3(z)$: $Y(z) = a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + b_0 V_2(z)$

$$H_3(z) = \frac{Y(z)}{V_2(z)} = \frac{b_0}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

$a_1 = a_2 = 3$ $b_0 = 4$ $\rightarrow H_3(z) = \dfrac{4}{1 - 3z^{-1} - 3z^{-2}} = \dfrac{4z^2}{z^2 - 3z - 3}$

roots of $z^2 - 3z - 3 = 0$

$\rightarrow z = \frac{3}{2} \pm \frac{1}{2}\sqrt{21}$ since $H_3(z)$ has poles outside of unit circle, then the causal system is unstable.

b) $$H(z) = H_1(z) H_2(z) H_3(z) = \frac{4z^2 (z^{-2} + 5z^{-3})(1 + 3z^{-1})}{z^2 - 3z - 3}$$

c) $$H(z) = \frac{b_0 z^2 (z^{-2} + 5z^{-3})(1 + 3z^{-1})}{z^2 - a_1 z - a_2} = 10 z^{-(n_d - 1)} + 55 \frac{z^{-n_d}}{1 - \frac{1}{2} z^{-1}}$$

$$\rightarrow \frac{b_0 (1 + 5z^{-1})(1 + 3z^{-1})}{z^2 - a_1 z - a_2} = \frac{z^{-(n_d - 1)} (10z - 5 + 55)}{z - \frac{1}{2}}$$

① left side should have also pole at $z = \frac{1}{2} \Rightarrow (\frac{1}{2})^2 - a_1(\frac{1}{2}) - \frac{a}{2} = 0$ ①

② one of the zeros of leftside should cancel with a pole of leftside, so that left and right becom equivalent.

Note that rightside has a zero at $z = -5$

This $(1 + 3z^{-1})$ should be cancelled $\Rightarrow (-3)^2 - a_1(-3) - a_2 = 0$ ②

solving ① and ② $\Rightarrow a_1 = -2.5$ $a_2 = 1.5$

9.5 cont.

$$\frac{b_0 z^{-2}(z+5)(z+3)}{(z+3)(z-\frac{1}{2})} = \frac{10 z^{-(n_d-1)}(z+5)}{z-\frac{1}{2}}$$

$\Rightarrow b_0 = 10 \qquad n_d = 3$

9.6

S_1, S_2, S_3, S_4 are IIR $\Longleftrightarrow$ #4, #2
S_5, S_6, S_7, S_8 are FIR $\Longleftrightarrow$ #1, #3

Using locations of pole and zero for S_1, S_2, S_3, S_4 we can conclude: $S_3 \Longleftrightarrow$ #2

$S_2 \Longleftrightarrow$ #4

using the fact that S_5, S_7 have two poles and two zero. while S_6, S_8 have 3 poles and 3 zeros, we conclude:

$S_5, S_7 \Longleftrightarrow$ #1

$S_6, S_8 \Longleftrightarrow$ #3

But using locations of zeros for plot 1 and plot 3, we conclude:

$S_7 \Longleftrightarrow$ #1

$S_6 \Longleftrightarrow$ #3

9.7

FIR: $J, L, N \Longleftrightarrow S_5, S_6, S_7, S_8$

IIR: $K, M, O \Longleftrightarrow S_1, S_2, S_3, S_4$

J has four nonzero coef. $\Longleftrightarrow S_6, S_8$

Since S_6 takes only ± 1 values $\Longrightarrow$

$$J \Longleftrightarrow S_8$$

Similarly, N has four nonzero ± 1 value, thus

$$N \Longleftrightarrow S_6$$

L has three nonzero values $\Longleftrightarrow S_5, S_7$

Since S_5 takes ± 1 values, we conclude

$$L \Longleftrightarrow S_7$$

Using the inverse z-transform, we conclude that

$$K \Longleftrightarrow S_4$$
$$M \Longleftrightarrow S_3$$
$$O \Longleftrightarrow S_2$$