GEORGIA INSTITUTE OF TECHNOLOGY

ECE 2025

SOLUTIONS Problem Set # 11

Spring 2006

PROBLEM 11.1:

(a) $x(t) = \delta(t+1) + 2\delta(t) + \delta(t-1) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \mathrm{e}^{j\omega} + 2 + \mathrm{e}^{-j\omega} = 2(1+\cos\omega) = 4\cos^2\frac{\omega}{2}$

(b) $x(t) = \frac{sin(100\pi(t-2))}{\pi(t-2)} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \mathrm{e}^{-j2\omega}[u(\omega+100\pi) - u(\omega-100\pi)]$

(c) $x(t) = \mathrm{e}^{-t}u(t) - \mathrm{e}^{-t}u(t-4) = \mathrm{e}^{-t}u(t) - \mathrm{e}^{-4}\,\mathrm{e}^{-(t-4)}u(t-4) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1+j\omega} - \mathrm{e}^{-4j\omega}\frac{\mathrm{e}^{-4}}{1+j\omega}$
$= \frac{1}{1+j\omega}\left(1 - \mathrm{e}^{-4(1+j\omega}\right)$

PROBLEM 11.2:

(a) $x(t)$ is the derivative of the function with transform $\frac{1}{0.1+j\omega}$, delayed by 0.2 time units. Thus, $x(t) = \frac{d}{dt}\left(\mathrm{e}^{-0.1(t-0.2)}u(t-0.2)\right) = -0.1\,\mathrm{e}^{-0.1(t-0.2)}u(t-0.2) + \mathrm{e}^{-0.1(t-0.2)}\delta(t-0.2)$. This is, $x(t) = -0.1\,\mathrm{e}^{-0.1(t-0.2)}u(t-0.2) + \delta(t-0.2)$.

(b) $X(j\omega) = 2 + 2\cos\omega = 2 + \mathrm{e}^{j\omega} + \mathrm{e}^{-j\omega} \overset{\mathcal{F}}{\longleftrightarrow} 2\delta(t) + \delta(t+1) + \delta(t-1)$.

(c) $X(j\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega} \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{e}^{-t}u(t) - e^{-2t}u(t)$.

(d) $X(j\omega) = j\delta(\omega - 100\pi) - j\delta(\omega+100\pi) \overset{\mathcal{F}}{\longleftrightarrow} \frac{j}{2\pi}\mathrm{e}^{j100\pi t} - \frac{j}{2\pi}e^{-j100\pi t} = -\frac{1}{\pi}\sin 100\pi t$.

PROBLEM 11.3:
Guess the reference is to Fig. P-11.11

(a) $x(t) = \sum_{k=-\infty}^{\infty} a_k\,\mathrm{e}^{jk200\pi t}$ has Fourier transform $X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - 200k\pi)$. Thus, with $a_k = \frac{1}{\pi k^2}$ for $k \neq 0$ and 1 for $k = 0$ you get $X(j\omega) = 2\pi\delta(\omega) + \sum_{k=1}^{\infty}\frac{2}{k^2}[\delta(\omega - 200k\pi) + \delta(\omega + 200k\pi)]$.

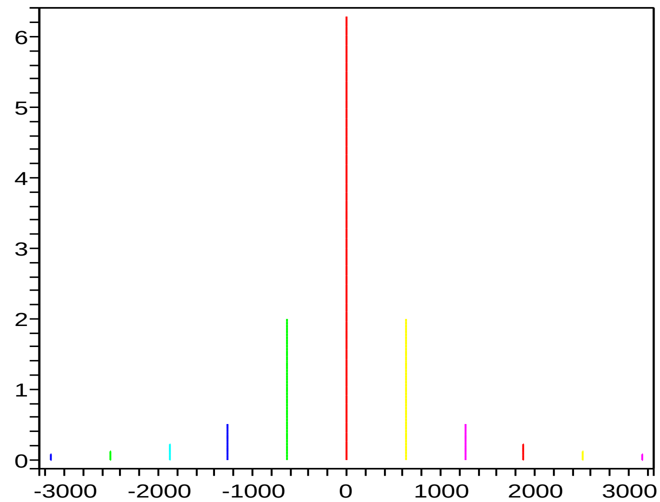
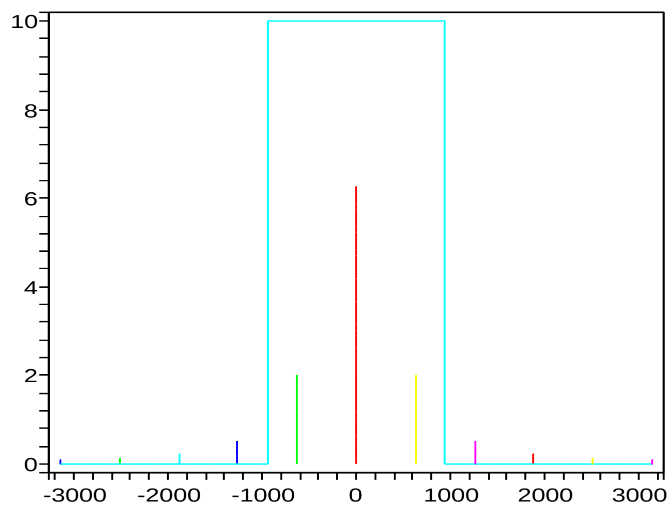


Figure 1: Spectrum of x



(b)

Figure 2: Spectrum with ideal filter

(c) DC value of output signal: $10\,\frac{1}{2\pi}2\pi = 10$.

(d) Expression for output

$$\begin{aligned} y(t) &= 10 + \mathcal{F}^{-1}[2\delta(\omega - 200\pi) + 2\delta(\omega + 200\pi)] \\ &= 10 + \frac{1}{\pi}[\,\mathrm{e}^{j200\pi t} + \mathrm{e}^{-j200\pi t}] \\ &= 10 + \frac{2}{\pi}\cos(200\pi t). \end{aligned}$$

PROBLEM 11.4:
A continuous-time LTI system is defined by the impulse response

$$h(t) = \delta(t) - b\,\mathrm{e}^{-bt}u(t)$$

(a) Determine the Fourier transform, $H(j\omega)$, which is also the frequency response of the system. Express your answer as a rational form with a simple numerator and denominator.

$$\begin{aligned}
H(j\omega) &= \int_{-\infty}^{\infty} h(t)\,\mathrm{e}^{-j\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \left[\delta(t) - b\,\mathrm{e}^{-bt}u(t)\right]\,\mathrm{e}^{-j\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \delta(t)\,\mathrm{e}^{-j\omega t}\mathrm{d}t - b\int_{0}^{\infty} \mathrm{e}^{-bt}\,\mathrm{e}^{-j\omega t}\,\mathrm{d}t \\
&= 1 - b\int_{0}^{\infty} \mathrm{e}^{-(b+j\omega)t}\,\mathrm{d}t \\
&= 1 + \frac{b}{(b+j\omega)}\left[\mathrm{e}^{-(b+j\omega)t}\right]_{0}^{\infty} \\
&= 1 - \frac{b}{(b+j\omega)} \\
&= \frac{j\omega}{(b+j\omega)}.
\end{aligned}$$

Note: Implicit assumption is that $b > 0$, otherwise the integral does not converge.

(b) Make a plot of the magnitude of the frequency response versus ω when $b = 200\pi$. The plot should cover the frequency range $0 \leq \omega < \infty$, but if you check your plot with MATLAB you will have to pick a maximum frequency, and that upper frequency should be at least ten times b.

The magnitude of the frequency response is

$$\left|\frac{j\omega}{(b+j\omega)}\right| = \frac{\omega}{\sqrt{b^2+\omega^2}}$$

For $\omega \to \infty$, the magnitude approaches 1 asymptotically (from below).

(c) Describe the type of filter in the plot of the previous part (e.g., LPF, HPF, or BPF). This is a HP filter. (low frequencies are attenuated, DC is completely blocked. High frequencies pass through with little attenuation).

(d) Determine the phase of $H(j\omega)$ at $\omega = 0, 200\pi$, and 1000π.
Note that the phase is given by $\frac{\pi}{2} - \arg(b+j\omega) = \frac{\pi}{2} - \arg(b+j\omega)$ for $\omega > 0$ and $-\frac{\pi}{2} - \arg(b+j\omega) = \frac{\pi}{2} - \arg(b+j\omega)$.

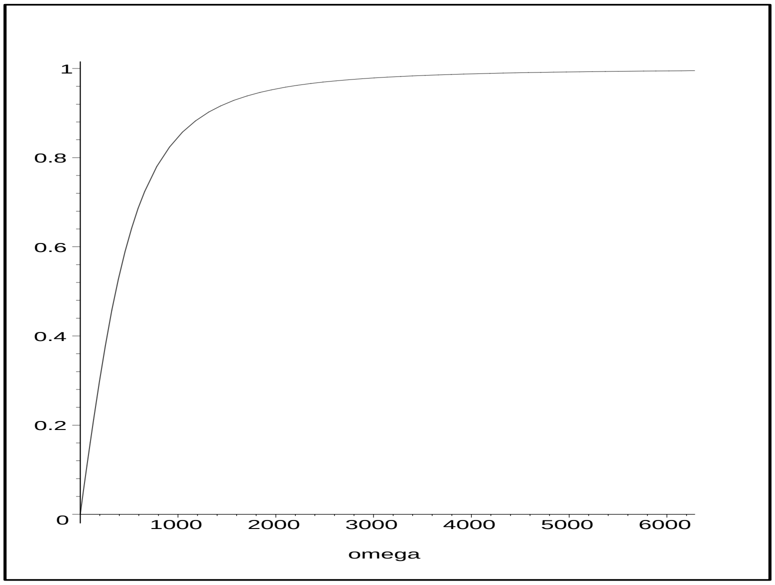


Figure 3: Magnitude plot

Hence, at $\omega = 0$, the phase is undetermined, but this does not matter as for $\omega = 0$, the response is zero anyway.
At $\omega = 200\pi = b$,

$$\Phi(200\pi) = \frac{\pi}{2} - \arg(200\pi + j200\pi) = \frac{\pi}{2} - \arg(1 + j) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

At $\omega = 1000\pi = b$, the phase in rad is

$$\Phi(1000\pi) = \frac{\pi}{2} - \arg(200\pi + j1000\pi) = \frac{\pi}{2} - \arg(1 + 5j) = \frac{\pi}{2} - \arctan(5).$$

Numerically, that's about $\frac{\pi}{2} - 1.373400767 = .197395560$.

(e) When the input signal is $x(t) = 10 + 20\cos(200\pi t + \pi/3) + 30\cos(1000\pi t)$, determine the output signal. Use the value of b given in part (b).
From the frequency response,

$$\begin{aligned}
y(t) &= |H(0)|10 + |H(j200\pi)|20\cos\left(200\pi t + \frac{\pi}{3} + \frac{\pi}{4}\right) + |H(j1000\pi)|30\cos(1000\pi t) \\
&= \frac{1}{\sqrt{2}}\,20\,\cos\left(200\pi t + \frac{7\pi}{12}\right) + \frac{1000\pi}{\sqrt{(200\pi)^2 + (1000\pi)^2}}\,30\,\cos(1000\pi t) \\
&= 14.14213562\,\cos\left(200\pi t + \frac{7\pi}{12}\right) + 29.41742027\,\cos(1000\pi t)\,.
\end{aligned}$$

PROBLEM 11.5*:
The impulse response of an LTI system is

$$h(t) = \cos(80\pi(t - 1/100))\frac{\sin(20\pi(t - 1/100))}{\pi(t - 1/100)}$$

(a) Determine the frequency response $H(j\omega)$ of the system which is an ideal filter.
(b) Make a sketch of the magnitude and phase of $H(j\omega)$ over the frequency range $-\infty < \omega < \infty$.
(c) Describe the type of filter in the plot of the previous part (e.g., LPF, HPF, or BPF).
(d) Using the filter defined above, determine the output of the system when the input signal is

$$x(t) = \cos(75\pi(t - 1/100)) + \frac{\sin(20\pi(t - 1/100))}{\pi(t - 1/100)}$$

Hint: Use frequency-domain methods: Determine the Fourier transform of the input signal, apply the filter in the frequency-domain to determine the Fourier transform of the output, and then do the inverse transform to get the corresponding output signal.

Solution
(a) Use the interesting properties, Note that $h(t) = g(t-1/100)$, where $g(t) = \cos(80\pi t)\frac{\sin(20\pi t)}{\pi t} = \cos(80\pi t)f(t)$, where $f(t) = \frac{\sin(20\pi t)}{\pi t}$. The Fourier transform of $f(t)$ is $F(j\omega) = 1$ for $|\omega| < 20\pi$, and zero else. By the modulation property, the transform of $g(t)$ is $\frac{1}{2}(F(j(\omega - 80\pi)) + F(j(\omega+80\pi))$. By the shift property, the transform of $h(t)$ is $H(j\omega) = \mathrm{e}^{-j\omega/100}G(j\omega)$. Putting it all together,

$$\begin{aligned} H(j\omega) &= \frac{\mathrm{e}^{-j\omega/100}}{2}\left[F(j(\omega - 80\pi)) + F(j(\omega + 80\pi))\right] \\ &= \begin{cases} \frac{1}{2}\,\mathrm{e}^{-j\omega/100} & \text{if} \quad 60\pi < \omega < 100\pi \\ \frac{1}{2}\,\mathrm{e}^{j|\omega|/100} & \text{if} \quad -100\pi < \omega < -60\pi \\ 0 & \text{else} \end{cases} \end{aligned}$$

(b) The phase is $-\omega/100$ in the pass bands. The phase is immaterial elsewhere.
(c) This is a BP filter.
(d) The input has Fourier transform

$$X(j\omega) = \mathrm{e}^{-j\omega/100}\frac{1}{2}[\delta(\omega - 75\pi) + \delta(\omega + 75\pi)] + e^{-j\omega/100}F(j\omega)$$

with $F(j\omega)$ as above. Note that $\pm 75\pi$ falls within the pass band, but all frequencies in $F(j\omega)$ are rejected. Hence

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) = \frac{1}{2}\,\mathrm{e}^{-j2\omega/100}\frac{1}{2}[\delta(\omega - 75\pi) + \delta(\omega + 75\pi)] \\ &= \frac{1}{4}\,\mathrm{e}^{-j150\pi/100}\delta(\omega - 75\pi) + \frac{1}{4}\,\mathrm{e}^{j150\pi/100}\delta(\omega + 75\pi) \end{aligned}$$

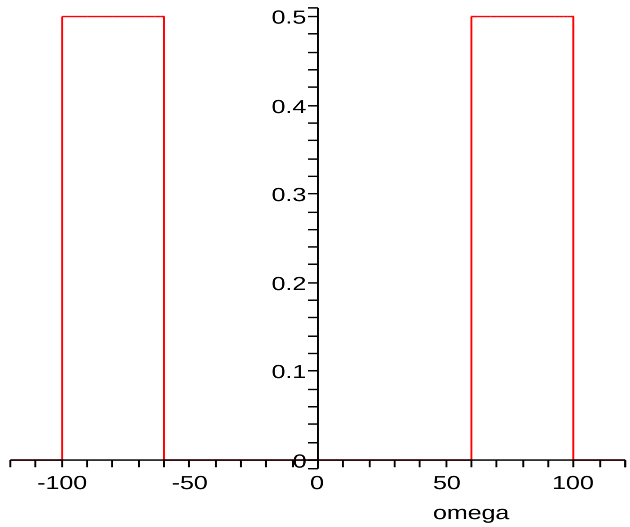


Figure 4: Magnitude plot

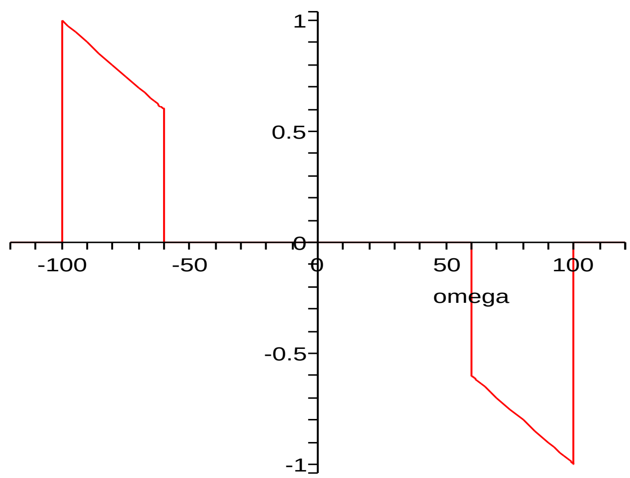


Figure 5: Phase plot

Thus, by the inverse Fourier transform,

$$\begin{aligned} y(t) &= \frac{1}{4}\,\mathrm{e}^{-j150\pi/100}\,\mathrm{e}^{j75\pi t} + \frac{1}{4}\,\mathrm{e}^{j150\pi/100}\,\mathrm{e}^{-j75\pi t} \\ &= \frac{1}{2}\cos(75\pi t - 3\pi/2). \end{aligned}$$