# Signal Processing First

**LECTURE #2**
**Phase & Time-Shift**
**Complex Exponentials**



# READING ASSIGNMENTS

- This Lecture:
  - Chapter 2, Sects. 2-3 to 2-5
- Appendix A: Complex Numbers
- Appendix B: MATLAB
- Next Lecture: finish Chap. 2,
  - Section 2-6 to end



# LECTURE OBJECTIVES

- Define Sinusoid Formula from a plot
- Relate TIME-SHIFT to PHASE

Introduce an **ABSTRACTION**:
Complex Numbers **represent** Sinusoids
Complex Exponential Signal

$$z(t) = Xe^{j\omega t}$$



# SINUSOIDAL SIGNAL

$$A\cos(\omega t + \varphi)$$

- **FREQUENCY** $\omega$
  - **Radians/sec**
  - or, Hertz (cycles/sec)

$$\omega = (2\pi) f$$

- **PERIOD** (in sec)

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

- **AMPLITUDE** $A$
  - Magnitude
- **PHASE** $\varphi$

# PLOTTING COSINE SIGNAL from the FORMULA

$$5\cos(0.3\pi t + 1.2\pi)$$

- Determine **period**:

$$T = 2\pi/\omega = 2\pi/0.3\pi = 20/3$$

- Determine a **peak** location by solving

$$(\omega t + \varphi) = 0$$

- **Peak at t=-4**

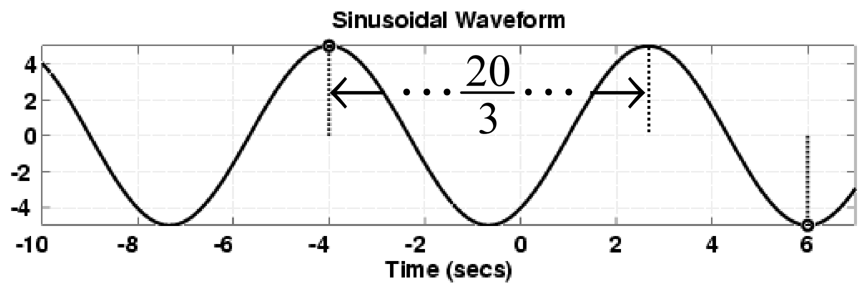




# ANSWER for the PLOT

$$5\cos(0.3\pi t + 1.2\pi)$$

- Use T=20/3 and the peak location at t=-4




# TIME-SHIFT

- In a mathematical formula we can replace t with $t-t_m$

$$x(t - t_m) = A\cos(\omega(t - t_m))$$

- Then the t=0 point moves to $t=t_m$

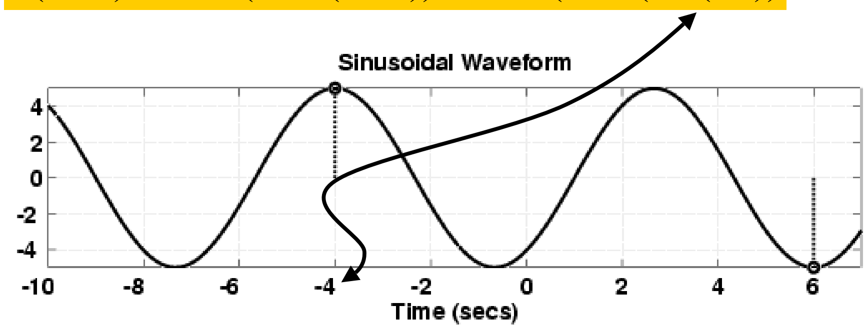
- Peak value of $\cos(\omega(t-t_m))$ is now at $t=t_m$





# TIME-SHIFTED SINUSOID

$$x(t+4) = 5\cos(0.3\pi(t+4)) = 5\cos(0.3\pi(t-(-4)))$$



# PHASE <--> TIME-SHIFT

- Equate the formulas:

$$A\cos(\omega(t-t_m)) = A\cos(\omega t + \varphi)$$

- and we obtain: $$-\omega t_m = \varphi$$
- or, $$t_m = -\frac{\varphi}{\omega}$$



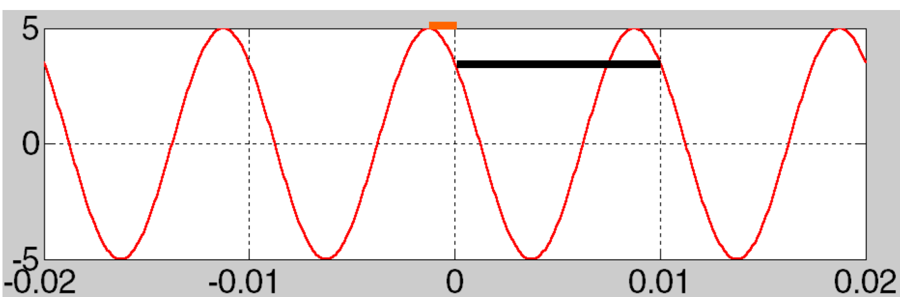

# SINUSOID from a PLOT

- Measure the period, T
  - Between peaks or zero crossings
  - Compute frequency: $\omega = 2\pi/T$
- Measure time of a peak: $t_m$
  - Compute phase: $\phi = -\omega t_m$
- Measure height of positive peak: A

3 steps




# (A, ω, ϕ) from a PLOT

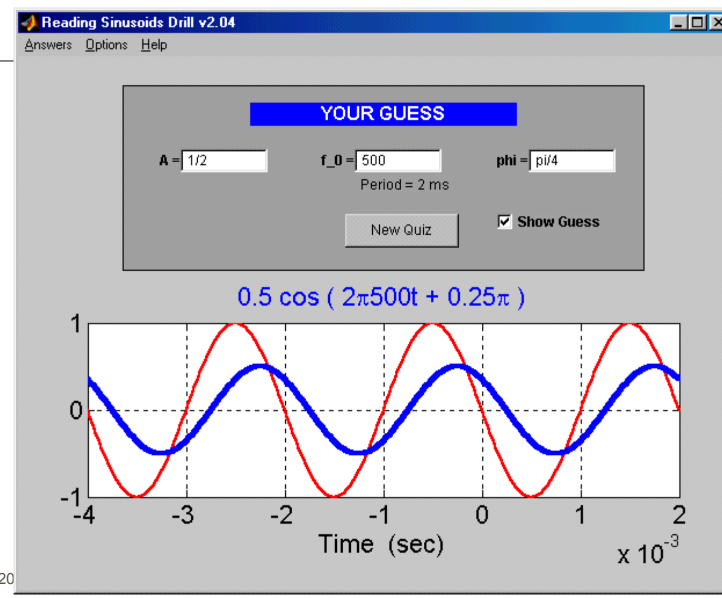



$$T = \frac{0.01\,\text{sec}}{1\,\text{period}} = \frac{1}{100} \rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{0.01} = 200\pi$$

$$t_m = -0.00125\,\text{sec} \rightarrow \varphi = -\omega t_m = -(200\pi)(t_m) = 0.25\pi$$




# SINE DRILL (MATLAB GUI)



# PHASE is AMBIGUOUS

- The cosine signal is periodic
  - Period is $2\pi$

$$A\cos(\omega t+\varphi+2\pi)=A\cos(\omega t+\varphi)$$

  - Thus adding any multiple of $2\pi$ leaves x(t) unchanged

$$\text{if } t_m=\frac{-\varphi}{\omega}, \text{ then}$$

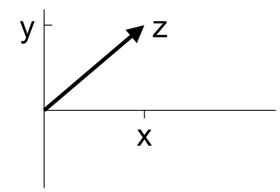
$$t_{m_2}=\frac{-(\varphi+2\pi)}{\omega}=\frac{-\varphi}{\omega}-\frac{2\pi}{\omega}=t_m-T$$





# COMPLEX NUMBERS

- To solve: $z^2 = -1$
  - z = ***j***
  - Math and Physics use z = ***i***
- Complex number: z = x + ***j*** y

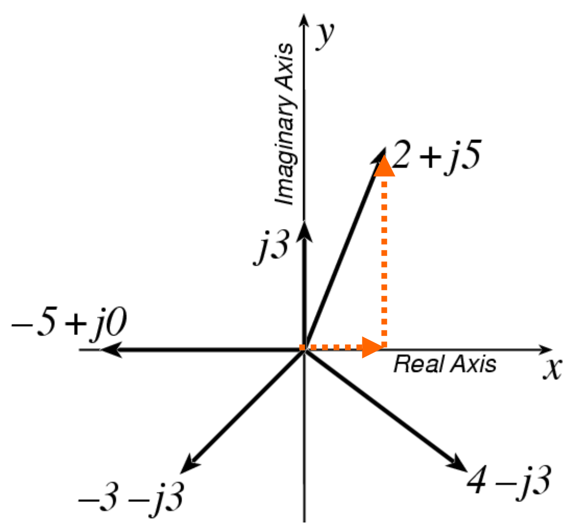



Cartesian coordinate system





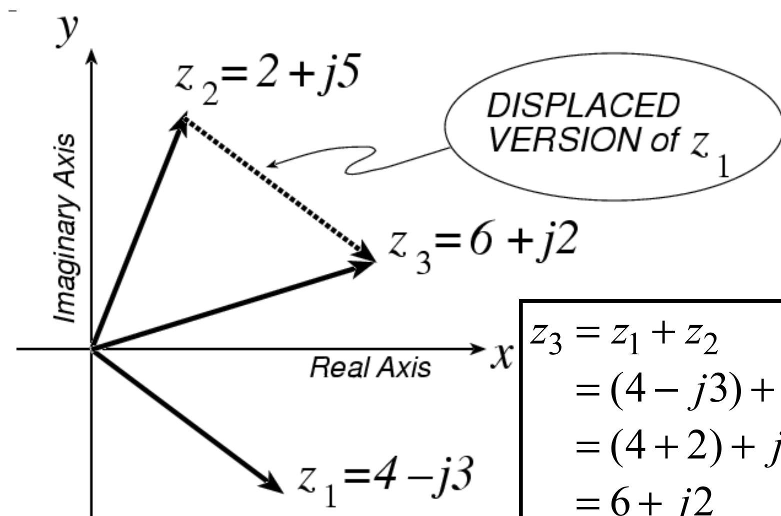
# PLOT COMPLEX NUMBERS







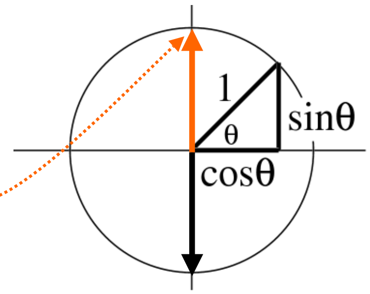
# COMPLEX ADDITION = VECTOR Addition




$$\begin{aligned} z_3 &= z_1 + z_2 \\ &= (4 - j3) + (2 + j5) \\ &= (4+2) + j(-3+5) \\ &= 6 + j2 \end{aligned}$$

# *** POLAR FORM ***

- Vector Form
  - **Length** =1
  - **Angle** = $\theta$
- Common Values
  - **j** has angle of $0.5\pi$
  - $-1$ has angle of $\pi$
  - **– j** has angle of $1.5\pi$
  - also, angle of **–j could** be $-0.5\pi = 1.5\pi - 2\pi$
  - because the PHASE is **AMBIGUOUS**








# POLAR <--> RECTANGULAR

- Relate (x,y) to (r,θ)

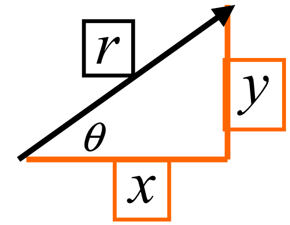
$$r^2 = x^2 + y^2$$

$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$$

Most calculators do Polar-Rectangular

$$x = r\cos\theta$$

$$y = r\sin\theta$$



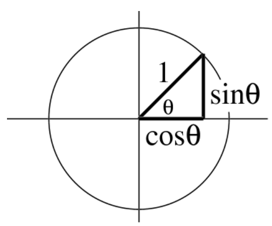

***Need a notation for POLAR FORM***





# Euler's FORMULA

- **Complex Exponential**
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one




$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

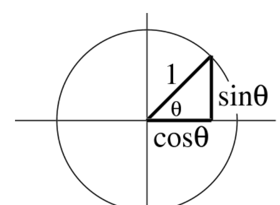




# COMPLEX EXPONENTIAL

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
  - $\theta = \omega t$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates $0.2\pi$ in 0.01 secs




$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

# cos = REAL PART

Real Part of Euler's

$$\cos(\omega t) = \Re e\{e^{j\omega t}\}$$

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi)$$

So,

$$A\cos(\omega t + \varphi) = \Re e\{Ae^{j(\omega t + \varphi)}\}$$

$$= \Re e\{Ae^{j\varphi}e^{j\omega t}\}$$



# REAL PART EXAMPLE

$$A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Evaluate:

$$x(t) = \Re e\left\{-3je^{j\omega t}\right\}$$

Answer:

$$x(t) = \Re e\left\{(-3j)e^{j\omega t}\right\}$$

$$= \Re e\left\{3e^{-j0.5\pi}e^{j\omega t}\right\} = 3\cos(\omega t - 0.5\pi)$$



# COMPLEX AMPLITUDE

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

**Complex AMPLITUDE = X**

$$z(t) = Xe^{j\omega t} \qquad X = Ae^{j\varphi}$$

Then, any Sinusoid = REAL PART of $Xe^{j\omega t}$

$$x(t) = \Re e\left\{Xe^{j\omega t}\right\} = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$