Signal Processing First

Lecture 14
Z Transforms: Introduction

READING ASSIGNMENTS

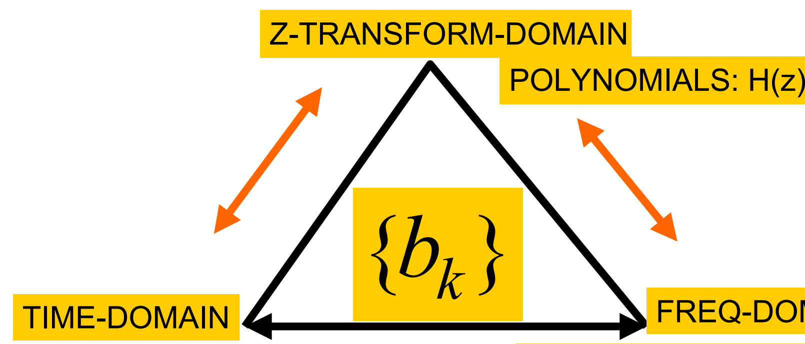
- This Lecture:
  - Chapter 7, Sects 7-1 through 7-5
- Other Reading:
  - Recitation: Ch. 7
    - CASCADING SYSTEMS
  - Next Lecture: Chapter 7, 7-6 to the end

LECTURE OBJECTIVES

- INTRODUCE the Z-TRANSFORM
  - Give Mathematical Definition
  - Show how the H(z) POLYNOMIAL simplifies analysis
    - CONVOLUTION is SIMPLIFIED !
- Z-Transform can be applied to
  - FIR Filter: $h[n] \rightarrow H(z)$
  - Signals: $x[n] \rightarrow X(z)$

$$H(z) = \sum_{n} h[n] z^{-n}$$

TWO (no, THREE) DOMAINS



$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

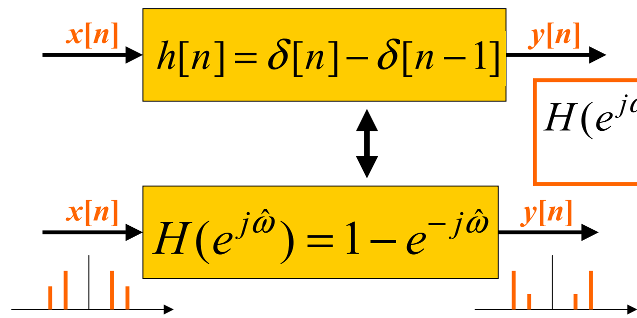
$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

TRANSFORM CONCEPT

- Move to a new domain where
  - OPERATIONS are EASIER & FAMILIAR
  - Use POLYNOMIALS
- TRANSFORM both ways
  - $x[n] \rightarrow X(z)$ (into the z domain)
  - $X(z) \rightarrow x[n]$ (back to the time domain)

"TRANSFORM" EXAMPLE

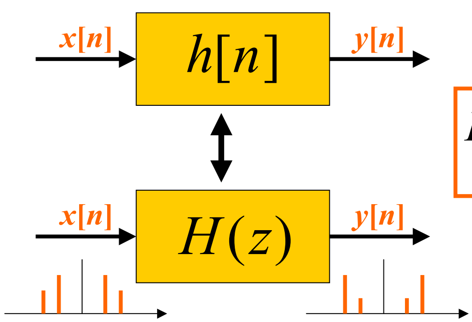
- Equivalent Representations



$$H(e^{j\hat{\omega}}) = \sum_n h[n] e^{-j\hat{\omega} n}$$

Z-TRANSFORM IDEA

- **POLYNOMIAL** REPRESENTATION



$$H(z) = \sum_n h[n] z^{-n}$$

Z-Transform DEFINITION

- POLYNOMIAL Representation of LTI SYSTEM: $H(z) = \sum_n h[n] z^{-n}$

APPLIES to Any SIGNAL

- EXAMPLE:

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

$$H(z) = 2z^{-0} + 0z^{-1} - 3z^{-2} + 0z^{-3} + 2z^{-4}$$

$$= 2 - 3z^{-2} + 2z^{-4}$$

$$= 2 - 3(z^{-1})^2 + 2(z^{-1})^4$$

POLYNOMIAL in z^{-1}

Z-Transform EXAMPLE

- ANY SIGNAL has a z-Transform:

$$X(z) = \sum_{n} x[n] z^{-n}$$

Example 7.1

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

$$X(z) = ?$$

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

Example 7.2

$$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$$

EXPONENT GIVES TIME LOCATION

$$x[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -2 & n = 1 \\ 0 & n = 2 \\ 3 & n = 3 \\ 0 & n = 4 \\ -1 & n = 5 \\ 0 & n > 5 \end{cases}$$

$$x[n] = ?$$

$$x[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-3] - \delta[n-5]$$

Z-Transform of FIR Filter

- CALLED the **SYSTEM FUNCTION**
  - h[n] is same as $\{b_k\}$

SYSTEM FUNCTION

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k] x[n-k]$$

FIR DIFFERENCE EQUATION

CONVOLUTION

Z-Transform of FIR Filter

- Get H(z) DIRECTLY from the $\{b_k\}$
- Example 7.3 in the book:

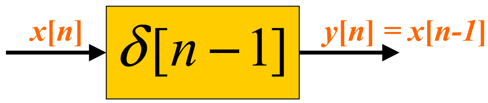
$$y[n] = 6x[n] - 5x[n-1] + x[n-2]$$

$$\{b_k\} = \{6, -5, 1\}$$

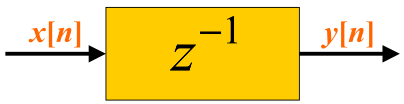
$$H(z) = \sum b_k z^{-1} = 6 - 5z^{-1} + z^{-2}$$

Ex. DELAY SYSTEM

- UNIT DELAY: find h[n] and H(z)



$$H(z) = \sum \delta[n-1] z^{-n} = z^{-1}$$



DELAY EXAMPLE

- UNIT DELAY: find y[n] via polynomials
  - x[n] = {3,1,4,1,5,9,0,0,0,...}

$$Y(z) = z^{-1} X(z)$$

$$Y(z) = z^{-1}(3 + z^{-1} + 4z^{-2} + z^{-3} + 5z^{-4} + 9z^{-5})$$

$$Y(z) = 0z^{0} + 3z^{-1} + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5} + 9z^{-6}$$

n	$n < 0$	0	1	2	3	4	5	6	$n > 6$
$y[n]$	0	0	3	1	4	1	5	9	0

DELAY PROPERTY

A delay of one sample multiplies the z-transform by z^{-1}.

$$x[n-1] \quad \Longleftrightarrow \quad z^{-1} X(z)$$

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n-n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$

GENERAL I/O PROBLEM

- Input is x[n], find y[n]   (for FIR, h[n])
- How to combine X(z) and H(z) ?

Example 7.5

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$

and $$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

and $$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

FIR Filter = CONVOLUTION

x[n], X(z)	0	+1	-1	+1	-1			
h[n], H(z)	1	2	3	4				
	0	+1	-1	+1	-1			
		0	+2	-2	+2	-2		
			0	+3	-3	+3	-3	
				0	+4	-4	+4	-4
y[n], Y(z)	0	+1	+1	+2	+2	-3	+1	-4

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

CONVOLUTION

CONVOLUTION PROPERTY

- PROOF:

$$y[n] = x[n] * h[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

$$Y(z) = \sum_{k=0}^{M} h[k]\left(z^{-k}X(z)\right)$$

$$= \left(\sum_{k=0}^{M} h[k]z^{-k}\right) X(z) = H(z)X(z).$$

MULTIPLY Z-TRANSFORMS

CONVOLUTION EXAMPLE

- **MULTIPLY** the z-TRANSFORMS:

Example 7.5

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$

and $h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

and $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

MULTIPLY $H(z)X(z)$

CONVOLUTION EXAMPLE

- Finite-Length input x[n]
- FIR Filter (L=4)

MULTIPLY Z-TRANSFORMS

$$Y(z) = H(z)X(z)$$

$$= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(z^{-1} - z^{-2} + z^{-3} - z^{-4})$$

$$= z^{-1} + (-1+2)z^{-2} + (1-2+3)z^{-3} + (-1+2-3+4)z^{-4}$$

$$+ \ (-2+3-4)z^{-5} + (-3+4)z^{-6} + (-4)z^{-7}$$

$$= z^{-1} + z^{-2} + 2z^{-3} + 2z^{-4} - 3z^{-5} + z^{-6} - 4z^{-7}$$

$y[n] = ?$

CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - Remember: $h_1[n] * h_2[n]$
  - How to combine $H_1(z)$ and $H_2(z)$?

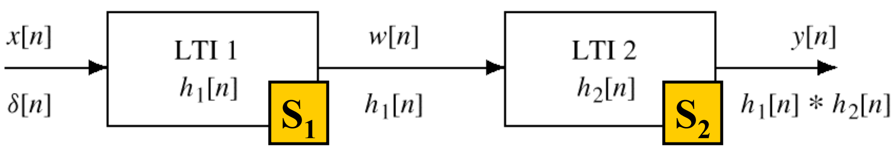
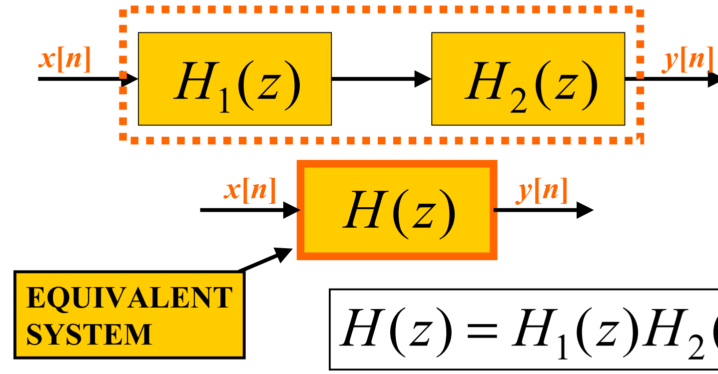


Figure 5.19 A Cascade of Two LTI Systems.

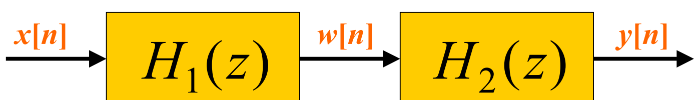
CASCADE EQUIVALENT

- Multiply the System Functions



$$H(z) = H_1(z)H_2(z)$$

CASCADE EXAMPLE



$$w[n] = x[n] - x[n-1]$$

$$y[n] = w[n] + w[n-1]$$

$$H_1(z) = 1 - z^{-1}$$

$$H_2(z) = 1 + z^{-1}$$

$$H(z) = (1 - z^{-1})(1 + z^{-1}) = 1 - z^{-2}$$

$$y[n] = x[n] - x[n-2]$$