Signal Processing First

Lecture 17
IIR Filters: H(z) and
Frequency Response

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, Sects. 8-4 8-5 & 8-6
- Other Reading:
  - Recitation: Chapter 8, all
    - POLE-ZERO PLOTS

LECTURE OBJECTIVES

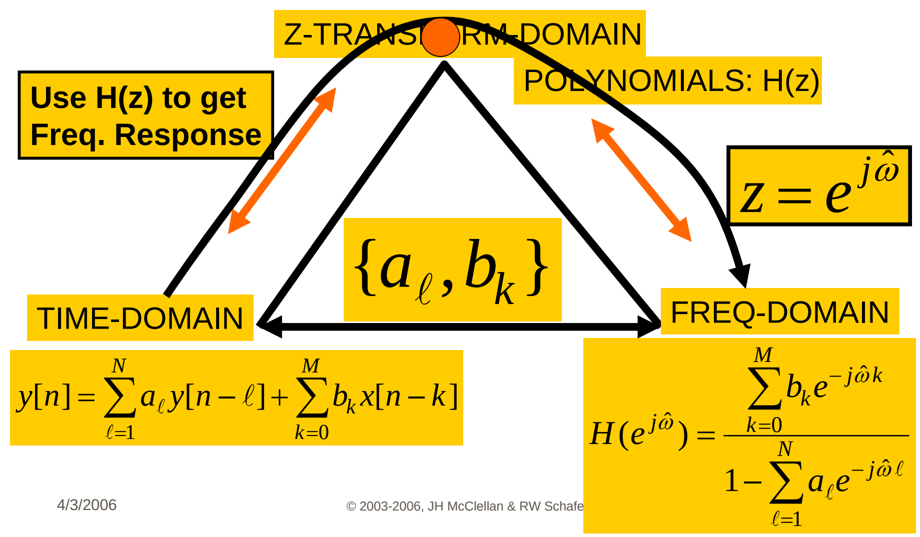
- SYSTEM FUNCTION: H(z)
- H(z) has **POLES** and ZEROS
- FREQUENCY RESPONSE of IIR
  - Get H(z) first

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

THREE DOMAINS



H(z) = z-Transform{ h[n] }

- FIRST-ORDER IIR FILTER:

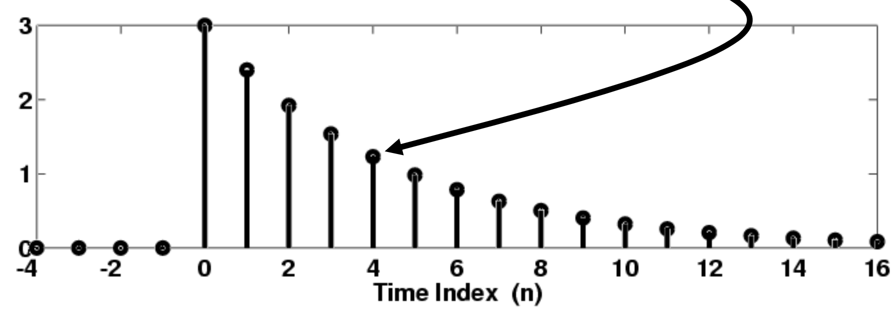
$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

Typical IMPULSE Response

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



First-Order Transform Pair

$$h[n] = b a^n u[n] \quad \leftrightarrow \quad H(z) = \frac{b}{1 - a z^{-1}}$$

- GEOMETRIC SEQUENCE:

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

DELAY PROPERTY of X(z)

- DELAY in TIME<-->Multiply X(z) by z^{-1}

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1} X(z)$$

Proof: $$\sum_{n=-\infty}^{\infty} x[n-1] z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-(\ell+1)}$$

$$= z^{-1} \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell} = z^{-1} X(z)$$

Z-Transform of IIR Filter

- DERIVE the SYSTEM FUNCTION H(z)
  - Use **DELAY** PROPERTY

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

EASIER with DELAY PROPERTY

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$

SYSTEM FUNCTION of IIR

- NOTE the FILTER COEFFICIENTS

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{B(z)}{A(z)}$$

SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$y[n] = 0.8 y[n-1] + 3x[n] - 2x[n-1]$$

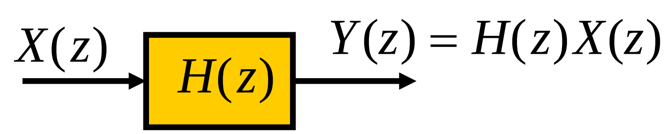
- **READ** the FILTER COEFFS:

H(z)

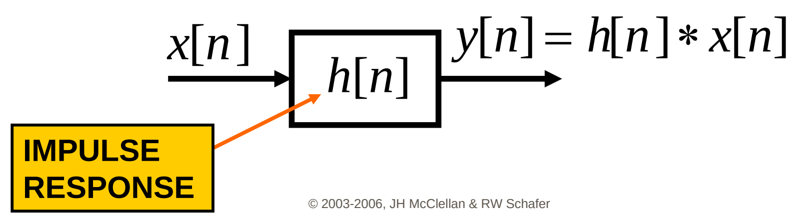
$$Y(z) = \left(\frac{3 - 2z^{-1}}{1 - 0.8 z^{-1}} \right) X(z)$$

CONVOLUTION PROPERTY

- **MULTIPLICATION** of z-TRANSFORMS



- **CONVOLUTION** in TIME-DOMAIN



POLES & ZEROS

- ROOTS of Numerator & Denominator

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$

ZERO: H(z)=0

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$

POLE: H(z) → inf

EXAMPLE: Poles & Zeros

- VALUE of H(z) at POLES is **INFINITE**

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

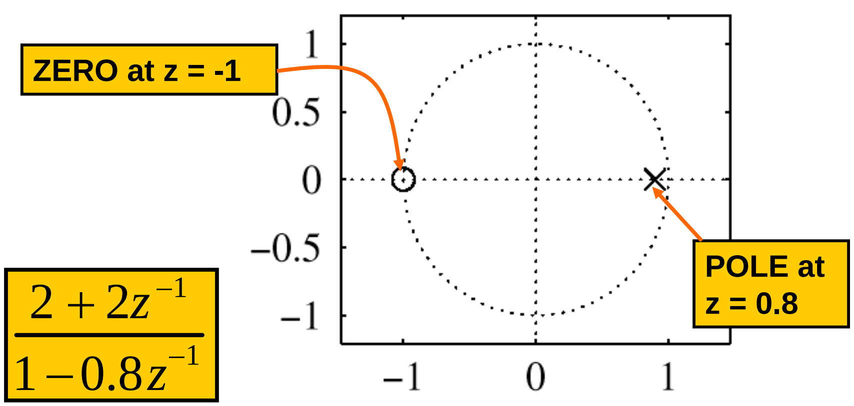
$$H(z) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

ZERO at z= -1

$$H(z) = \frac{2 + 2(\frac{4}{5})^{-1}}{1 - 0.8(\frac{4}{5})^{-1}} = \frac{\frac{9}{2}}{0} \rightarrow \infty$$

POLE at z=0.8

POLE-ZERO PLOT



$$\frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

FREQUENCY RESPONSE

- SYSTEM FUNCTION: H(z)
- H(z) has DENOMINATOR
- FREQUENCY RESPONSE of IIR
  - We have H(z)

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z = e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

FREQ. RESPONSE FORMULA

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}}$$

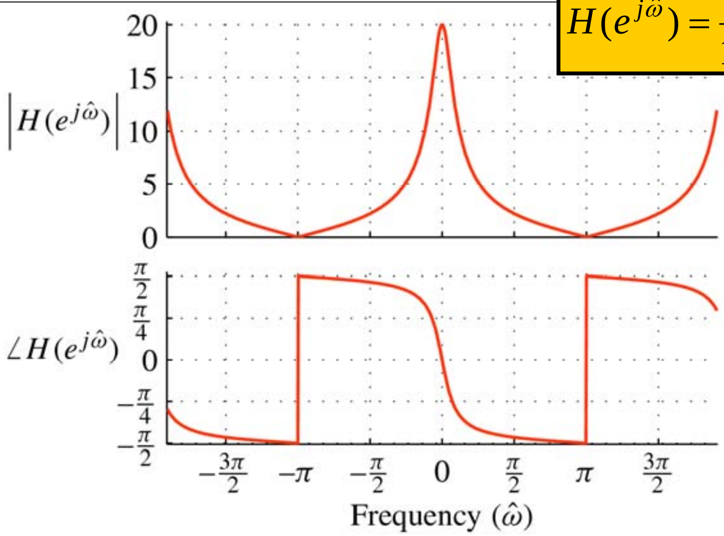
$$\left|H(e^{j\hat{\omega}})\right|^2 = \left|\frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}}\right|^2 = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}} \cdot \frac{2 + 2e^{j\hat{\omega}}}{1 - 0.8e^{j\hat{\omega}}}$$

$$\frac{4 + 4 + 4e^{-j\hat{\omega}} + 4e^{j\hat{\omega}}}{1 + 0.64 - 0.8e^{-j\hat{\omega}} - 0.8e^{j\hat{\omega}}} = \frac{8 + 8\cos\hat{\omega}}{1.64 - 1.6\cos\hat{\omega}}$$

$$@\,\hat{\omega} = 0, \quad \left|H(e^{j\hat{\omega}})\right|^2 = \frac{8+8}{0.04} = 400, \quad @\,\hat{\omega} = \pi\,?$$

Frequency Response Plot

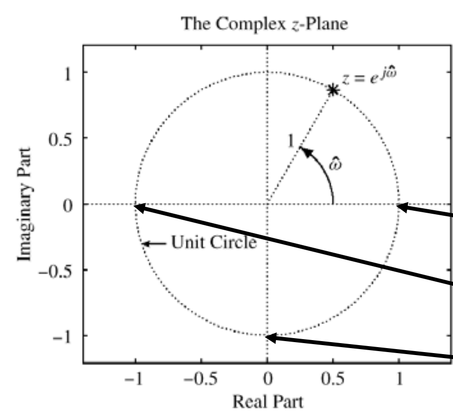
$$H(e^{j\hat{\omega}}) = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}}$$



UNIT CIRCLE

- MAPPING BETWEEN z and $\hat{\omega}$

$$z = e^{j\hat{\omega}}$$

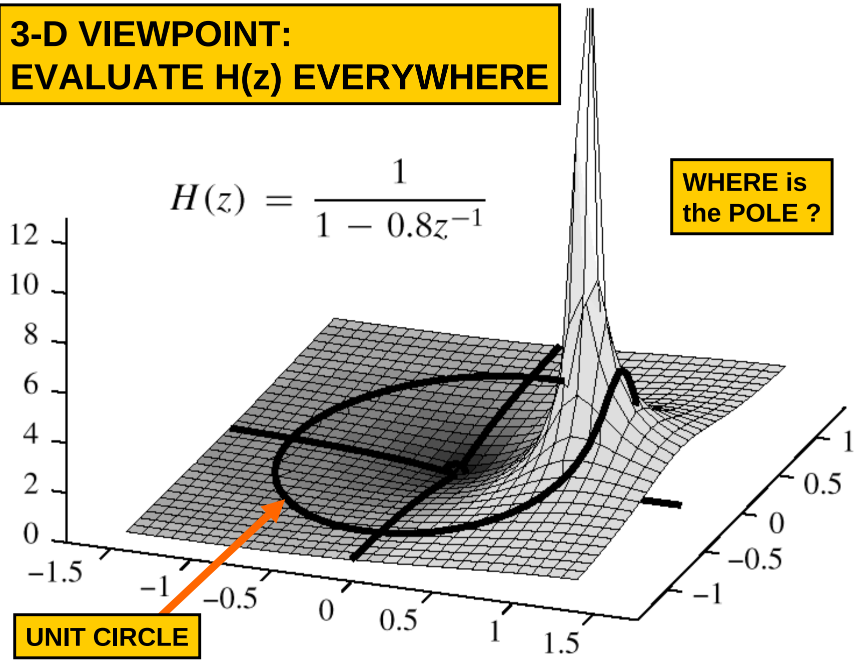


$z = 1 \quad \leftrightarrow \quad \hat{\omega} = 0$

$z = -1 \quad \leftrightarrow \quad \hat{\omega} = \pm\pi$

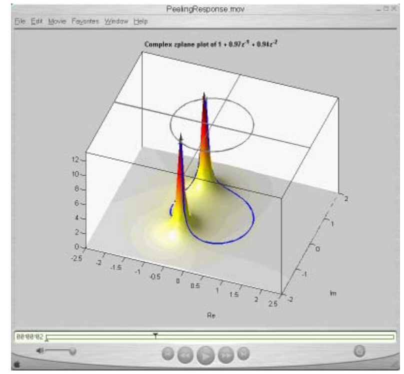
$z = \pm j \quad \leftrightarrow \quad \hat{\omega} = \pm\frac{1}{2}\pi$

3-D VIEWPOINT: EVALUATE H(z) EVERYWHERE



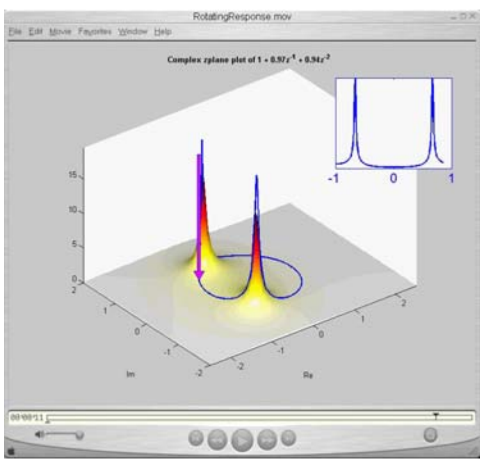
MOVIE for H(z) in 3-D

- POLES to H(z) to Frequency Reponse
  - TWO POLES SHOWN

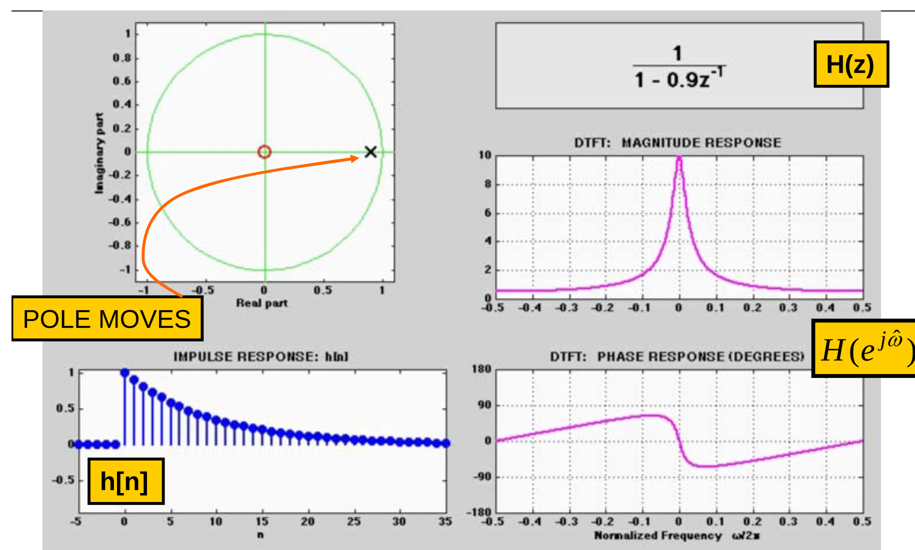


Frequency Response from H(z)

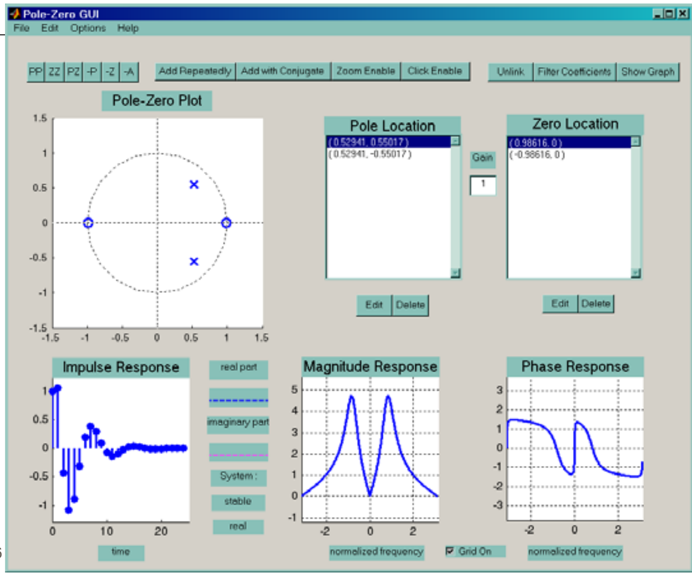
Walking around the Unit Circle



3 DOMAINS MOVIE: IIR



PeZ Demo: Pole-Zero Placing



SINUSOIDAL RESPONSE

- x[n] = SINUSOID => y[n] is SINUSOID
- Get MAGNITUDE & PHASE from H(z)

$$\text{if} \quad x[n] = e^{j\hat{\omega} n}$$

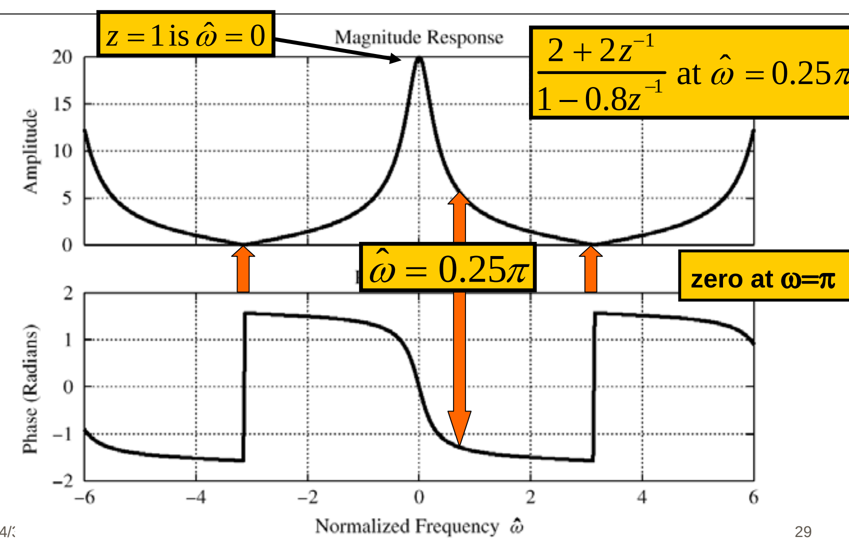
$$\text{then} \quad y[n] = H(e^{j\hat{\omega}})\, e^{j\hat{\omega} n}$$

$$\text{where} \quad H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

POP QUIZ

- Given: $H(z) = \dfrac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$
- Find the Impulse Response, h[n]
- Find the output, y[n]
  - When $x[n] = \cos(0.25\pi n)$

Evaluate FREQ. RESPONSE



POP QUIZ: Eval Freq. Resp.

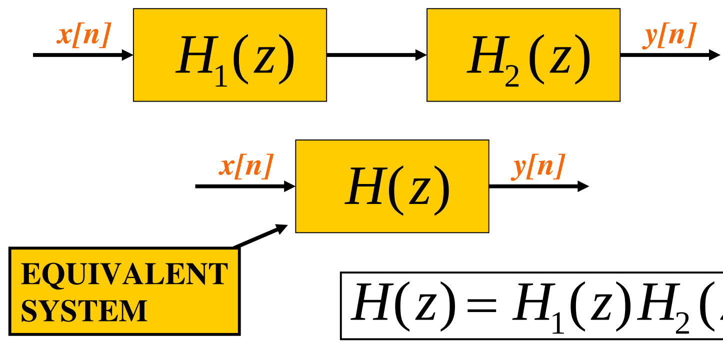
- Given: $H(z) = \dfrac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$
- Find output, y[n], when $x[n] = \cos(0.25\pi n)$
  - Evaluate at $z = e^{j0.25\pi}$

$$H(z) = \frac{2 + 2(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2})}{1 - 0.8e^{-j0.25\pi}} = 5.182e^{-j1.309}$$

$$y[n] = 5.182\cos(0.25\pi n - 0.417\pi)$$

CASCADE EQUIVALENT

- Multiply the System Functions



$$H(z) = H_1(z)H_2(z)$$