

# Signal Processing First

## Lecture 21 Frequency Response of Continuous-Time Systems

# READING ASSIGNMENTS

- This Lecture:
  - Chapter 10, all
- Other Reading:
  - Recitation: Ch. 10 all, start Ch 11
  - Next Lecture: Chapter 11

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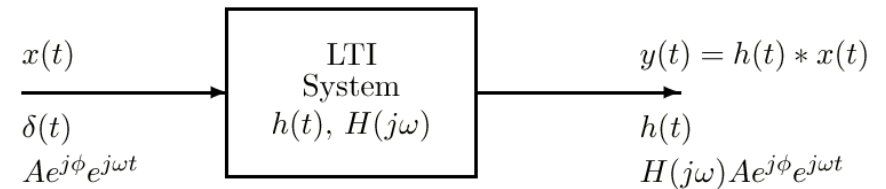
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# LECTURE OBJECTIVES

- Review of convolution
  - **THE** operation for **LTI** Systems
- Complex exponential input signals
  - **Frequency Response**
  - Cosine signals
    - Real part of complex exponential
- Fourier Series thru  $H(j\omega)$ 
  - These are Analog Filters

# LTI Systems



- Convolution defines an LTI system

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

- Response to a complex exponential gives frequency response  $H(j\omega)$

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## Thought Process #1

- **SUPERPOSITION (Linearity)**
  - Make  $x(t)$  a weighted sum of signals
  - Then  $y(t)$  is also a sum—same weights
    - But DIFFERENT OUTPUT SIGNALS usually
- Use **SINUSOIDS**
  - “SINUSOID IN GIVES SINUSOID OUT”
  - Make  $x(t)$  a weighted sum of sinusoids
  - Then  $y(t)$  is also a sum of sinusoids
    - Different Magnitudes and Phase
- **LTI SYSTEMS: Sinusoidal Response**

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## Thought Process #2

- **SUPERPOSITION (Linearity)**
  - Make  $x(t)$  a weighted sum of signals
- Use **SINUSOIDS**
  - **Any  $x(t)$  = weighted sum of sinusoids**
  - **HOW? Use FOURIER ANALYSIS INTEGRAL**
    - **To find the weights from  $x(t)$**
- **LTI SYSTEMS:**
  - Frequency Response changes each sinusoidal component

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## Complex Exponential Input

$$x(t) = Ae^{j\varphi} e^{j\omega t} \mapsto y(t) = \boxed{H(j\omega)} Ae^{j\varphi} e^{j\omega t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) Ae^{j\varphi} e^{j\omega(t-\tau)} d\tau$$

$$y(t) = \left( \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \right) Ae^{j\varphi} e^{j\omega t}$$

$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau}$$

**Frequency Response**

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## When does $H(j\omega)$ Exist?

- When is  $\boxed{|H(j\omega)| < \infty}$  ?

$$|H(j\omega)| = \left| \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau)| |e^{-j\omega\tau}| d\tau$$

$$\boxed{|H(j\omega)| \leq \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty}$$

- Thus the frequency response exists if the LTI system is a **stable** system.

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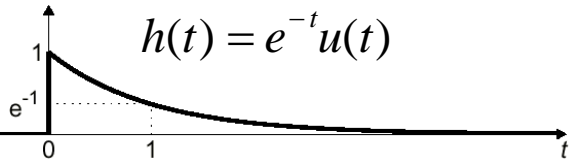
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$$h(t) = e^{-at} u(t) \Leftrightarrow H(j\omega) = \frac{1}{a + j\omega}$$

- Suppose that  $h(t)$  is:

$$a = 1$$



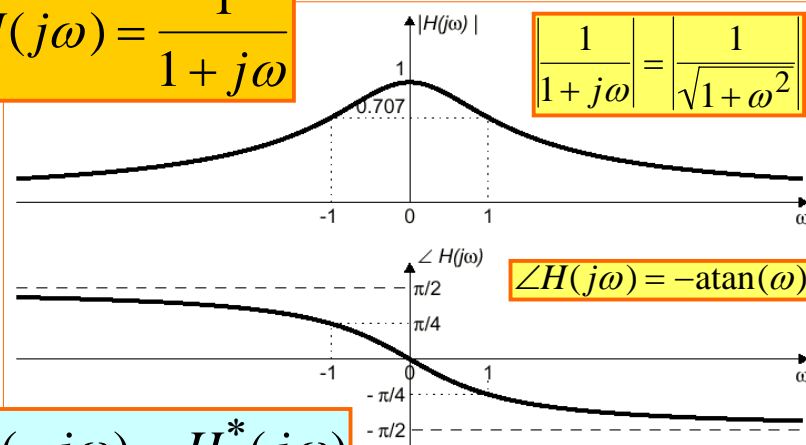
$$H(j\omega) = \int_{-\infty}^{\infty} e^{-a\tau} u(\tau) e^{-j\omega\tau} d\tau = \int_0^{\infty} e^{-(a+j\omega)\tau} d\tau$$

$$a > 0$$

$$H(j\omega) = \frac{e^{-(a+j\omega)\tau}}{-(a+j\omega)} \Big|_0^{\infty} = \frac{e^{-a\tau} e^{-j\omega\tau}}{-(a+j\omega)} \Big|_0^{\infty} = \frac{1}{a+j\omega}$$

## Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{1 + j\omega}$$



$$H(-j\omega) = H^*(j\omega)$$

## Freq Response of Integrator?

- Impulse Response
  - $h(t) = u(t)$
- NOT a Stable System
  - Frequency response  $H(j\omega)$  does NOT exist

$$h(t) = e^{-at} u(t) \Leftrightarrow H(j\omega) = \frac{1}{a + j\omega} \rightarrow \frac{1}{j\omega} ?$$

Need another term

“Leaky” Integrator (a is small)  
Cannot build a perfect Integral

$$a \rightarrow 0$$

## Ideal Delay: $y(t) = x(t - t_d)$

$$H(j\omega) = \int_{-\infty}^{\infty} \delta(\tau - t_d) e^{-j\omega\tau} d\tau = e^{-j\omega t_d}$$

$$H(j\omega) = e^{-j\omega t_d}$$

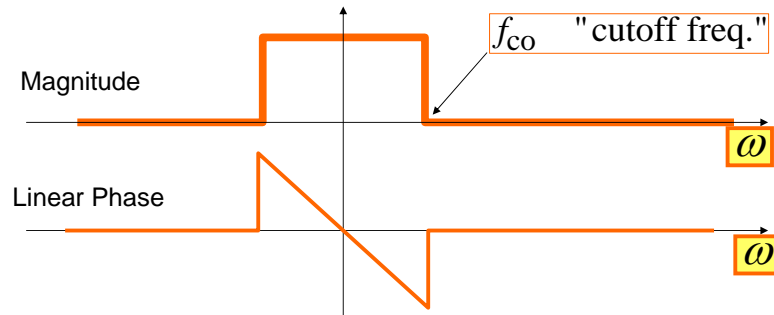
$$x(t) = e^{j\omega t} \mapsto$$

$$y(t) = e^{j\omega(t-t_d)} = \left( e^{-j\omega t_d} \right) e^{j\omega t}$$

$$H(j\omega)$$

# Ideal Lowpass Filter w/ Delay

$$H_{LP}(j\omega) = \begin{cases} e^{-j\omega t_d} & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



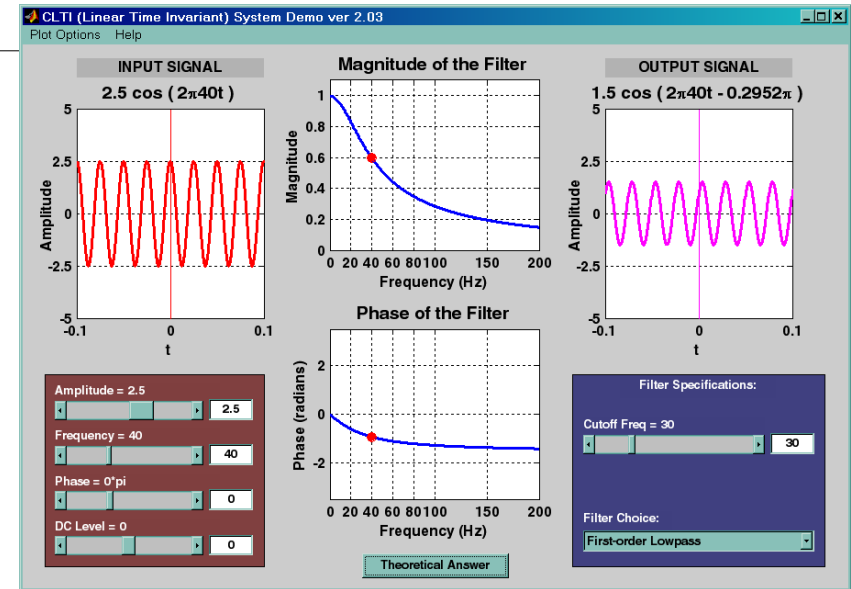
# Example: Ideal Low Pass

$$H_{LP}(j\omega) = \begin{cases} e^{-j3\omega} & |\omega| < 2 \\ 0 & |\omega| > 2 \end{cases}$$

$$x(t) = 10e^{j\pi/3} e^{j1.5t} \mapsto y(t) = H(j1.5)10e^{j\pi/3} e^{j1.5t}$$

$$y(t) = (e^{-j4.5})10e^{j\pi/3} e^{j1.5t} = 10e^{j\pi/3} e^{j1.5(t-3)}$$

# Sinusoid in Gives Sinusoid out



# Cosine Input

$$x(t) = A \cos(\omega_0 t + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 t} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 t}$$

$$y(t) = H(j\omega_0) \frac{A}{2} e^{j\phi} e^{j\omega_0 t} + H(-j\omega_0) \frac{A}{2} e^{-j\phi} e^{-j\omega_0 t}$$

$$\text{Since } H(-j\omega_0) = H^*(j\omega_0)$$

$$y(t) = A |H(j\omega_0)| \cos(\omega_0 t + \phi + \angle H(j\omega_0))$$

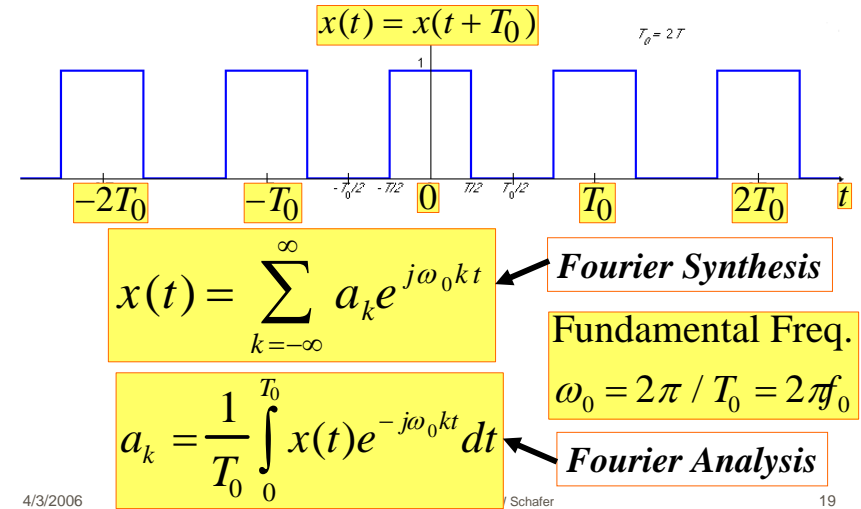
# Review Fourier Series

## ANALYSIS

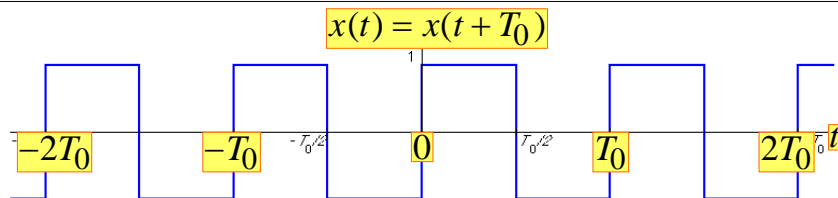
- Get representation from the signal
  - Works for PERIODIC Signals
- ## Fourier Series
- INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 kt} dt$$

# General Periodic Signals



# Square Wave Signal

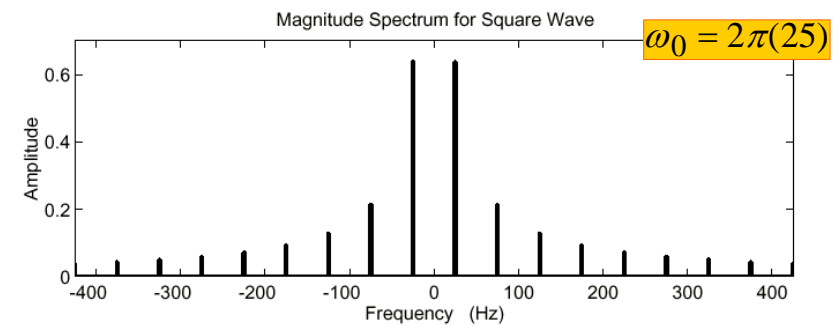


$$a_k = \frac{1}{T_0} \int_0^{T_0/2} (1) e^{-j\omega_0 kt} dt + \frac{1}{T_0} \int_{T_0/2}^{T_0} (-1) e^{-j\omega_0 kt} dt$$

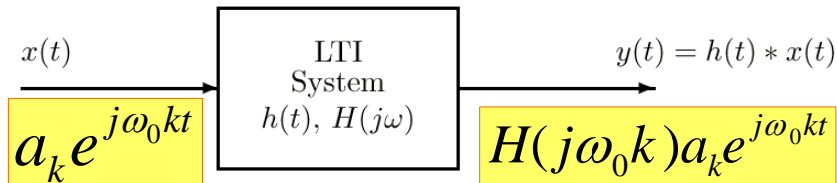
$$a_k = \frac{e^{-j\omega_0 kt} \Big|_0^{T_0/2}}{-j\omega_0 k T_0} - \frac{e^{-j\omega_0 kt} \Big|_{T_0/2}^{T_0}}{-j\omega_0 k T_0} = \frac{1 - e^{-j\pi k}}{j\pi k}$$

# Spectrum from Fourier Series

$$a_k = \frac{1 - e^{-j\pi k}}{j\pi k} = \begin{cases} \frac{2}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = 0, \pm 2, \pm 4, \dots \end{cases}$$



# LTI Systems with Periodic Inputs



By superposition,

Output has same frequencies

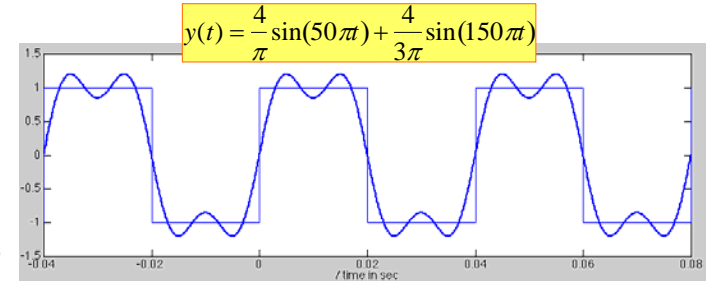
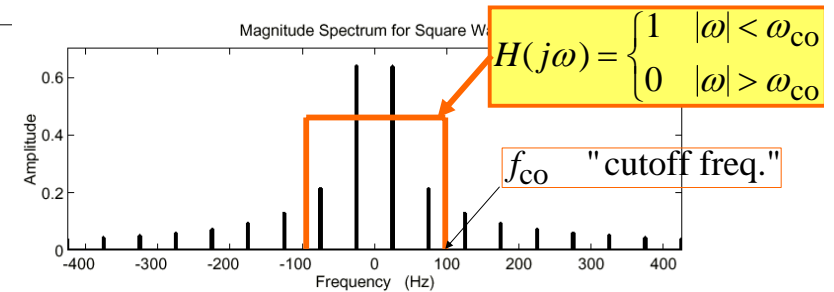
$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 k t} = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 k t}$$

$$b_k = a_k H(j\omega_0 k)$$

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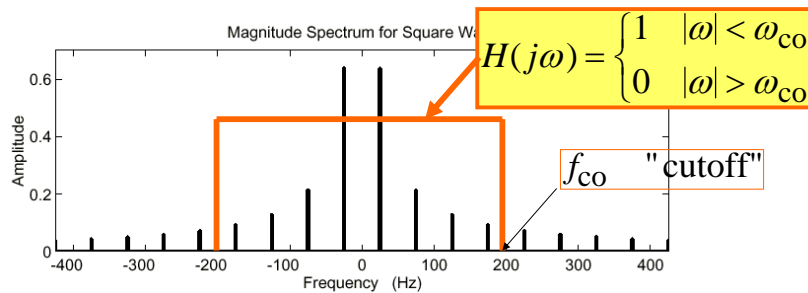
# Ideal Lowpass Filter (100 Hz)



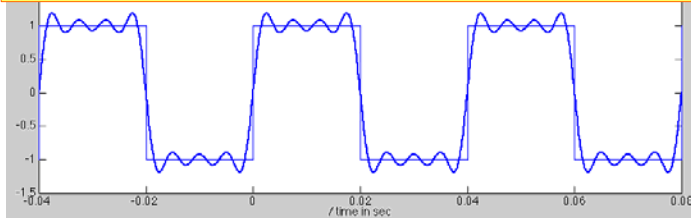
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# Ideal Lowpass Filter (200 Hz)



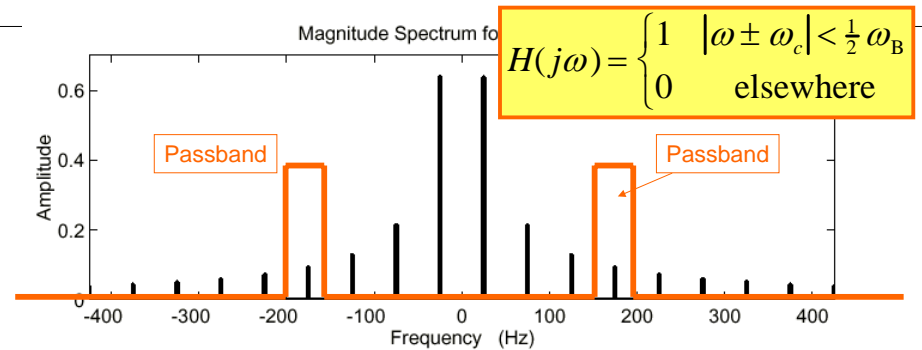
$$y(t) = \frac{4}{\pi} \sin(50\pi t) + \frac{4}{3\pi} \sin(150\pi t) + \frac{4}{5\pi} \sin(250\pi t) + \frac{4}{7\pi} \sin(350\pi t)$$



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# Ideal Bandpass Filter



What is the output signal ?

$$y(t) = \frac{2}{j7\pi} e^{j2\pi(175)t} - \frac{2}{j7\pi} e^{-j2\pi(175)t} = \frac{4}{7\pi} \cos(2\pi(175)t - \frac{1}{2}\pi)$$

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Example

$$H(j\omega) = e^{-j\omega t_d}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t} \mapsto y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 k t}$$

$$b_k = a_k H(j\omega_0 k) = a_k e^{-j\omega_0 k t_d}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{-j\omega_0 k t_d} e^{j\omega_0 k t} = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k (t-t_d)}$$

$$\therefore y(t) = x(t - t_d)$$