Signal Processing First

Lecture 25
Sampling and Reconstruction
(Fourier View)

LECTURE OBJECTIVES

- Sampling Theorem Revisited
  - GENERAL: in the FREQUENCY DOMAIN
  - Fourier transform of sampled signal
  - Reconstruction from samples
- Reading: Chap 12, Section 12-3
- Review of FT properties
  - Convolution ←→ multiplication
  - Frequency shifting
  - Review of AM

Table of FT Properties

$$x(t)*h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

Delay Property

$$x(t-t_d) \Leftrightarrow e^{-j\omega t_d}X(j\omega)$$

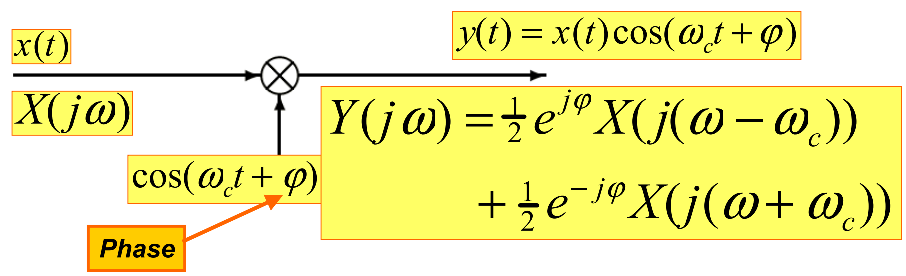
Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

Scaling

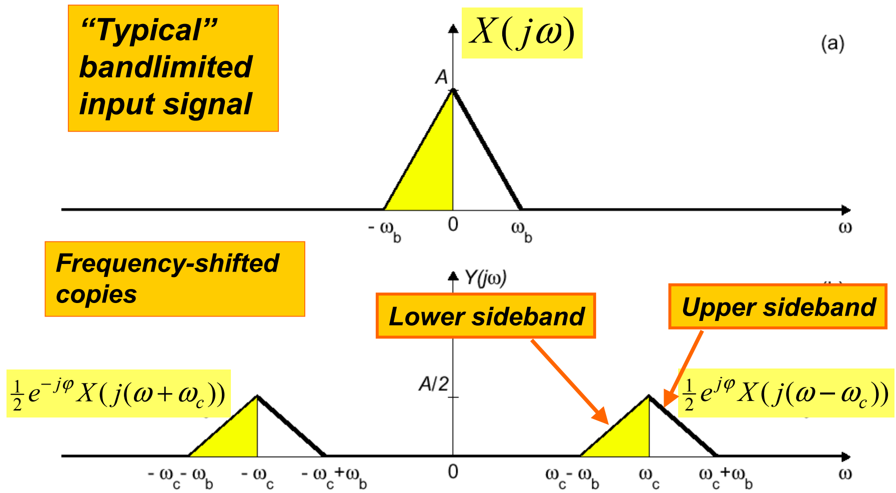
$$x(at) \Leftrightarrow \frac{1}{|a|}X(j(\frac{\omega}{a}))$$

Amplitude Modulator

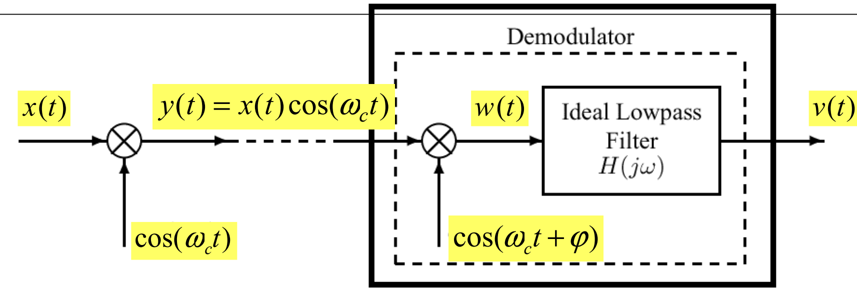


- *x(t)* modulates the amplitude of the cosine wave. The result in the frequency-domain is two SHIFTED copies of *X(jω)*.

DSBAM: Frequency-Domain



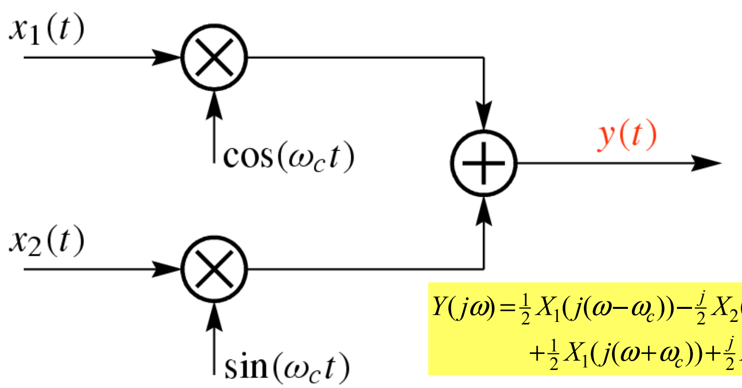
DSBAM Demod Phase Synch



$V(j\omega)=\frac{1}{2}\cos(\varphi)X(j\omega)$ what if $\varphi=\frac{1}{2}\pi$?

$$W(j\omega)=\frac{1}{4}e^{j\varphi}X(j\omega)+\frac{1}{4}e^{-j\varphi}X(j\omega)$$
$$+\frac{1}{4}e^{j\varphi}X(j(\omega-2\omega_c))+\frac{1}{4}e^{-j\varphi}X(j(\omega+2\omega_c))$$

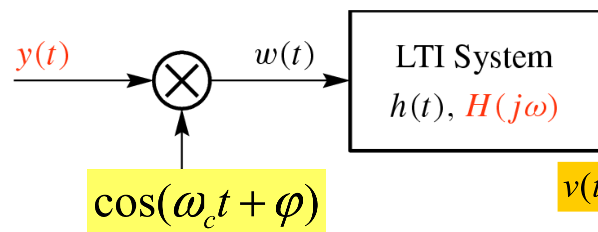
Quadrature Modulator



$$Y(j\omega)=\frac{1}{2}X_1(j(\omega-\omega_c))-\frac{j}{2}X_2(j(\omega-\omega_c))$$
$$+\frac{1}{2}X_1(j(\omega+\omega_c))+\frac{j}{2}X_2(j(\omega+\omega_c))$$

***TWO* signals on *ONE* channel: "out of phase"**
Can you "separate" them in the demodulator ?

Demod: Quadrature System



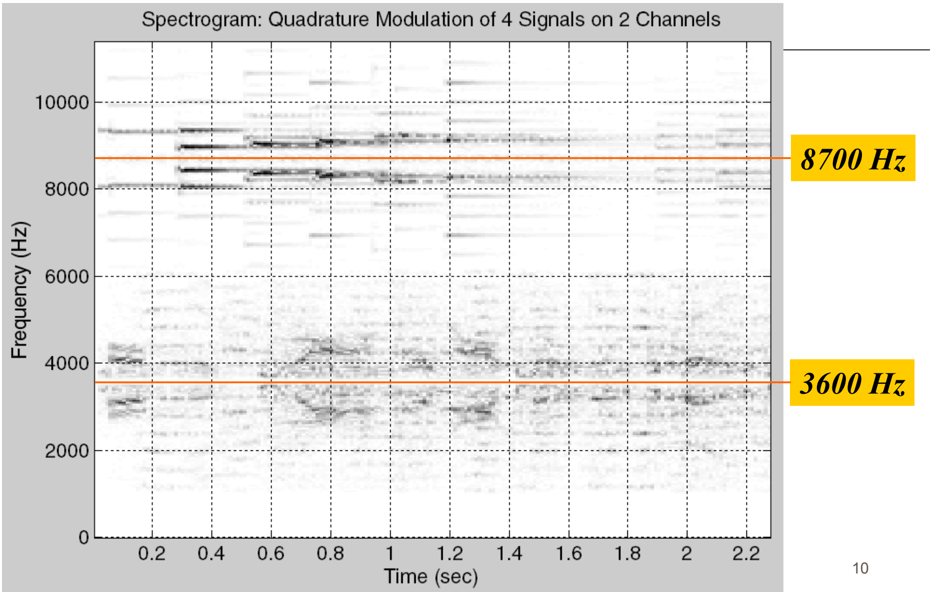
$v(t)=x_1(t)$ if $\varphi=0$

$v(t)=x_2(t)$ if $\varphi=-\pi/2$

$$Y(j\omega)=\frac{1}{2}X_1(j(\omega-\omega_c))-\frac{j}{2}X_2(j(\omega-\omega_c))$$
$$+\frac{1}{2}X_1(j(\omega+\omega_c))+\frac{j}{2}X_2(j(\omega+\omega_c))$$

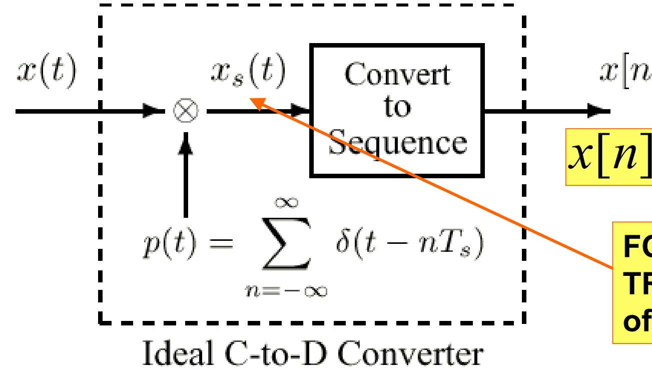
$$V(j\omega)=\frac{1}{4}e^{-j\varphi}X_1(j\omega)+\frac{1}{4}\underline{e^{-j\pi/2}e^{-j\varphi}}X_2(j\omega)+$$
$$\frac{1}{4}e^{j\varphi}X_1(j\omega)+\frac{1}{4}\underline{e^{j\pi/2}e^{j\varphi}}X_2(j\omega)$$

Quadrature Modulation: 4 sigs



Ideal C-to-D Converter

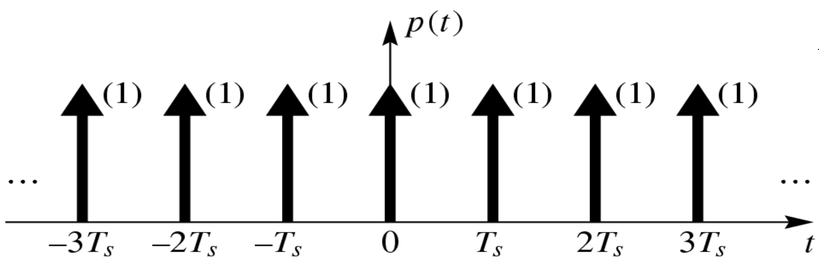
- **Mathematical Model for A-to-D**



$$x[n] = x(nT_s)$$

FOURIER TRANSFORM of $x_s(t)$???

Periodic Impulse Train



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

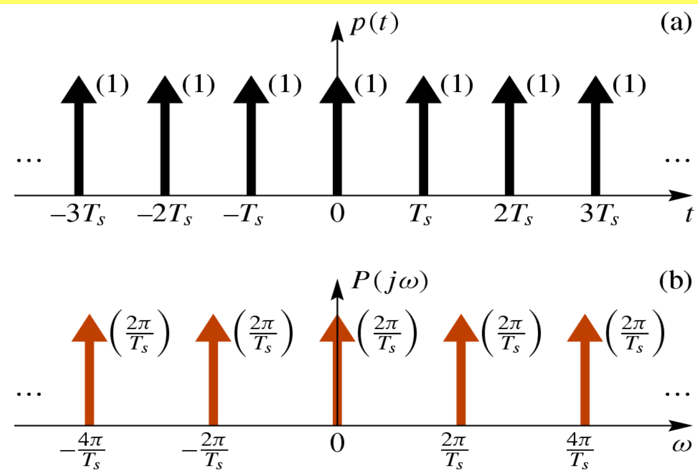
$$\omega_s = \frac{2\pi}{T_s}$$

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s}$$

Fourier Series

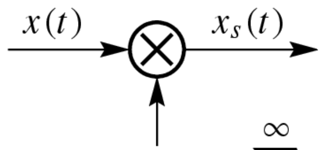
FT of Impulse Train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \leftrightarrow \quad P(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$



$$\omega_s = \frac{2\pi}{T_s}$$

Impulse Train Sampling

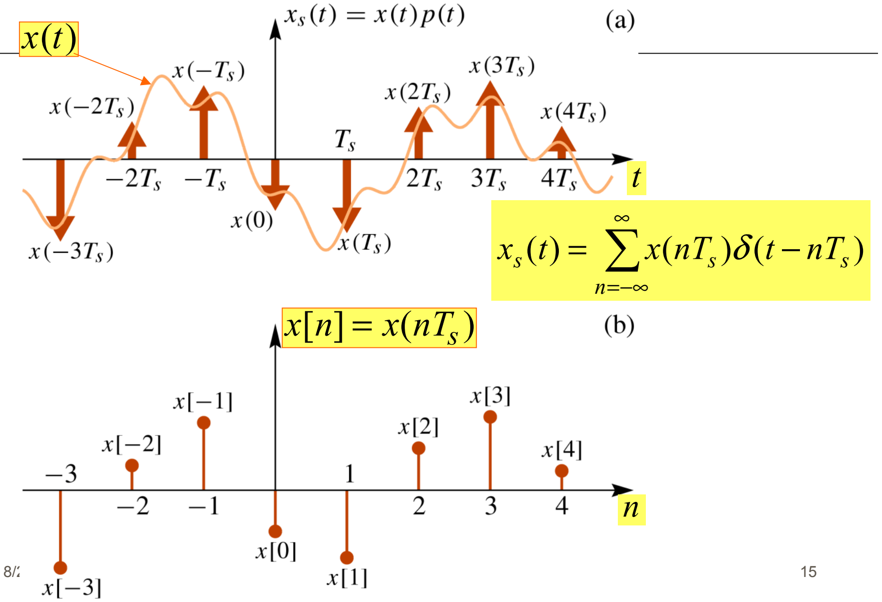


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

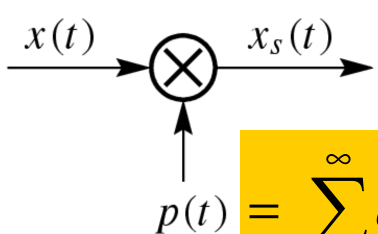
$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Illustration of Sampling



Sampling: Freq. Domain



$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

EXPECT FREQUENCY SHIFTING !!!

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

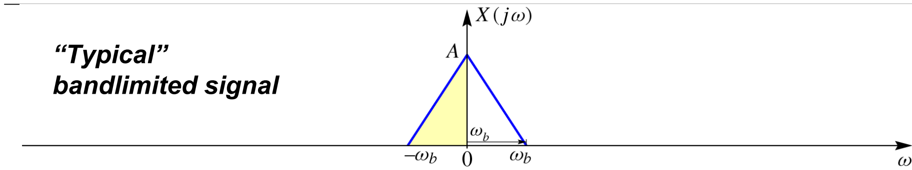
$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} x(t) e^{jk\omega_s t}$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

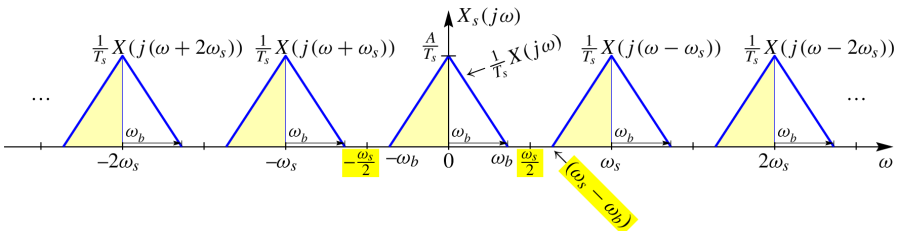
$$\omega_s = \frac{2\pi}{T_s}$$

Frequency-Domain Representation of Sampling

"Typical" bandlimited signal

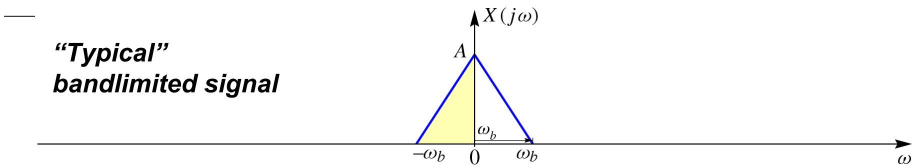


$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

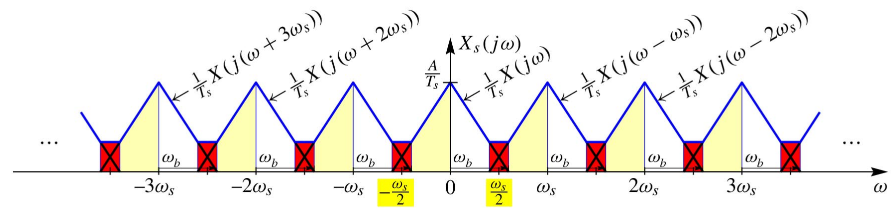


Aliasing Distortion

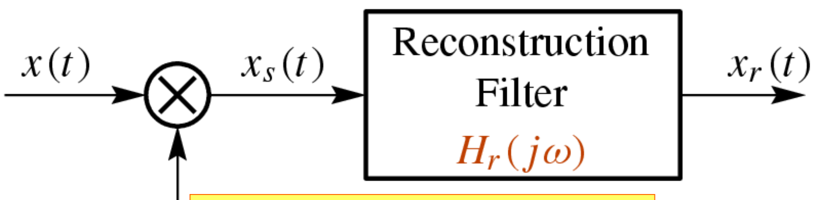
"Typical" bandlimited signal



- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have aliasing distortion.



Reconstruction of *x*(*t*)

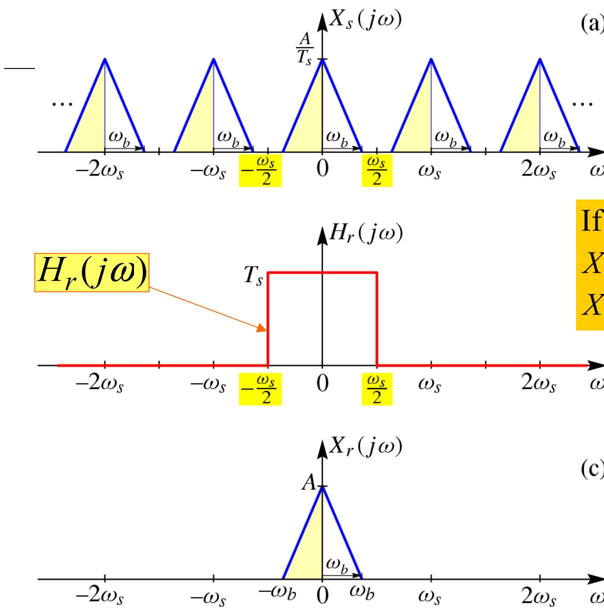


$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$

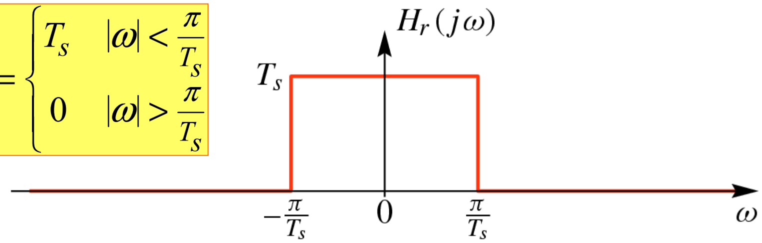
Reconstruction: Frequency-Domain



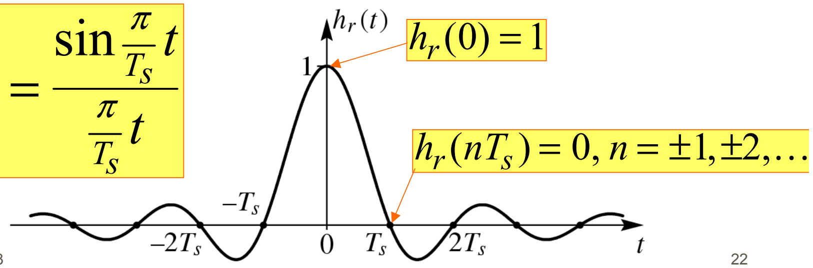
If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega)X_s(j\omega)$

Ideal Reconstruction Filter

$$H_r(j\omega)=\begin{cases} T_s & |\omega|<\frac{\pi}{T_s} \\ 0 & |\omega|>\frac{\pi}{T_s}\end{cases}$$



$$h_r(t)=\frac{\sin\frac{\pi}{T_s}t}{\frac{\pi}{T_s}t}$$



Signal Reconstruction

$$x_r(t)=h_r(t)*x_s(t)=h_r(t)*\sum_{n=-\infty}^{\infty}x(nT_s)\delta(t-nT_s)$$

$$x_r(t)=\sum_{n=-\infty}^{\infty}x(nT_s)h_r(t-nT_s)$$

$$x_r(t)=\sum_{n=-\infty}^{\infty}x(nT_s)\frac{\sin\frac{\pi}{T_s}(t-nT_s)}{\frac{\pi}{T_s}(t-nT_s)}$$

Ideal bandlimited interpolation formula

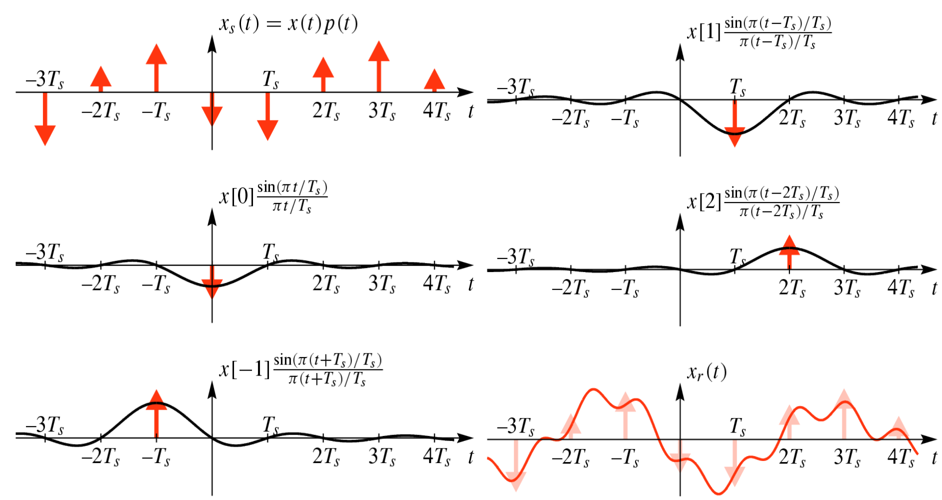
Shannon Sampling Theorem

- "SINC" Interpolation is the ideal
  - PERFECT RECONSTRUCTION
  - of BANDLIMITED SIGNALS

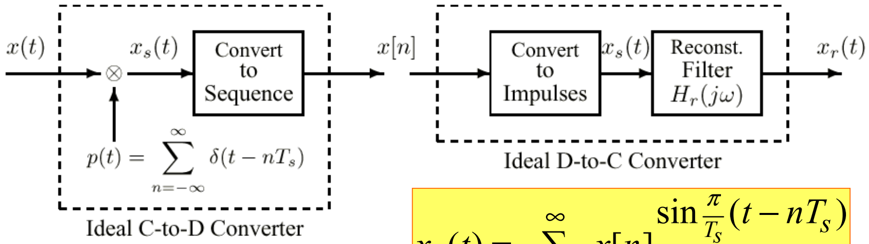
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega)=0$ for $|\omega|\geq\omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s=2\pi/T_s\geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t)=\sum_{n=-\infty}^{\infty}x(nT_s)\frac{\sin\left[\frac{\pi}{T_s}(t-nT_s)\right]}{\frac{\pi}{T_s}(t-nT_s)}.$$

Reconstruction in Time-Domain



Ideal C-to-D and D-to-C



$$x[n] = x(nT_s)$$

Ideal Sampler

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$