ECE-2025 Fall-2009

Lecture 4
More Complex Numbers
28-Aug-09

General Info

- **t-square** has all **<u>OFFICIAL</u>** msgs
- Skype Office Hours ?
- HW, Lab and MATLAB Help:
  - Mon, Tues, Wed: 6pm Klaus-2440
    - Weekly during the semester
- HW #1 solutions posted soon
- HW #3 posted
- H1N1? …email 2025 staff before absence

Lab Info

- Lab #1 Report (one per team)
  - Turn in at <u>beginning</u> of your lab time
  - **<u>OnLine Peer Evaluation</u>**, one for each student
    - Look under "Tests & Quizzes"
  - **Ask your grading TA about his/her format**
- Lab #2 will be posted soon
- Finish Instructor Verification in Lab
  - Come to lab PREPARED
- Computer Problems? **help@ece.gatech.edu**

POP QUIZ: Complex Amp

$$A\cos(\omega t+\varphi)=\Re\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Evaluate:

$$x(t)=\Re\left\{-3je^{j\omega t}\right\}$$

Answer:

$$x(t)=\Re\left\{(-3j)e^{j\omega t}\right\}$$

$$=\Re\left\{3e^{-j0.5\pi}e^{j\omega t}\right\}=3\cos(\omega t-0.5\pi)$$

POP QUIZ: Add Sinusoids

- ADD THESE 2 SINUSOIDS:

$$x_1(t) = \cos(4\pi(t - 0.25))$$

$$x_2(t) = \sqrt{2}\cos(4\pi(t + 0.75))$$

- COMPLEX (PHASOR) ADDITION:

$$1e^{-j\pi} + \sqrt{2}e^{j3\pi} = (1 + \sqrt{2})e^{j\pi}$$

$$x_1(t) + x_2(t) = (1 + \sqrt{2})\cos(4\pi t + \pi)$$

Lecture

READING ASSIGNMENTS

- This Lecture:
  - Appendix A: Complex Numbers
  - Chapter 2, Section 2-5
- Other Reading:
  - Next Lecture: Chapter 2, Section 2-6 to the end

LECTURE OBJECTIVES

- Introduce more tools for manipulating complex numbers
  - Conjugate
  - Multiplication & Division
  - Powers
  - N-th Roots of unity

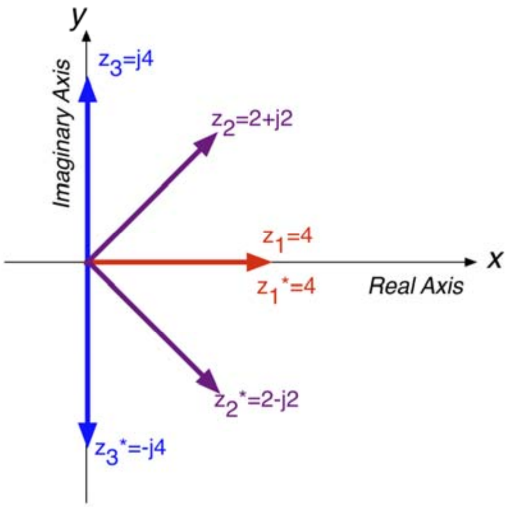
$$\text{For } z = e^{j2\pi k/N}, \quad z^N = 1$$

COMPLEX CONJUGATE

- Useful concept: change the sign of **all j's**
- **RECTANGULAR**: If z = x + ***j*** y, then the complex conjugate is z* = x – ***j*** y
- **POLAR**: Magnitude is the same but angle has sign change

$$z = re^{j\theta} \Rightarrow z^* = re^{-j\theta}$$

COMPLEX CONJUGATION



- *Flips vector about the real axis!*

USES OF CONJUGATION

- Conjugates useful for many calculations
- Real part:

$$\frac{z+z^*}{2} = \frac{(x+jy)+(x-jy)}{2} = x = \Re\{z\}$$

- Imaginary part:

$$\frac{z-z^*}{2j} = \frac{j2y}{2j} = y = \Im\{z\}$$

Inverse Euler Relations

- Cosine is real part of exp, sine is imaginary part
- Real part: $\frac{z+z^*}{2} = \Re\{z\}$

$$z = e^{j\theta}, \Rightarrow \Re\{e^{j\theta}\} = \frac{e^{j\theta}+e^{-j\theta}}{2} = \cos(\theta)$$

- Imaginary part:

$$\frac{z-z^*}{2j} = y = \Im\{z\}$$

$$z = e^{j\theta}, \Rightarrow \Im\{e^{j\theta}\} = \frac{e^{j\theta}-e^{-j\theta}}{2j} = \sin(\theta)$$

Mag & Magnitude Squared

- Magnitude Squared:

$$zz^* = (re^{j\theta})(re^{-j\theta}) = r^2 = |z|^2$$

- Magnitude Squared:

$$z\,z^* = (x+jy)\times(x-jy) = x^2 - j^2y^2 = x^2 + y^2$$

- Magnitude of complex exponential is one:

$$|e^{j\theta}|^2 = \cos^2(\theta) + \sin^2(\theta) = 1$$

MULTIPLICATION

- **CARTESIAN**: use polynomial algebra

$$z_1 \times z_2 = (x_1 + jy_1) \times (x_2 + jy_2)$$
$$= (x_1x_2 - y_1y_2) + j(x_1y_2 - y_1x_2)$$

- **POLAR**: easier because you can leverage the properties of exponentials

$$z_1 \times z_2 = r_1 e^{j\theta_1} \times r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

- Multiply the mags, Add the angles

DIVISION

- **CARTESIAN**: use complex conjugate to convert to multiplication:

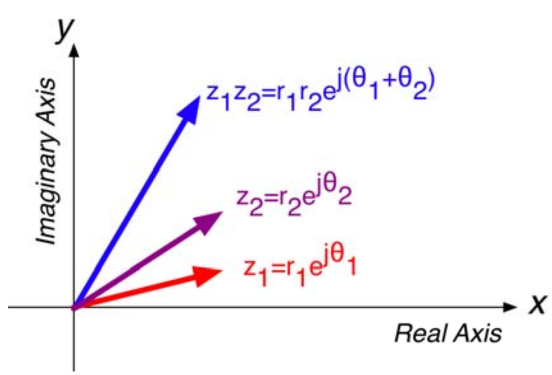
$$\frac{z_1}{z_2} = \frac{(x_1 + jy_1)}{(x_2 + jy_2)} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2}$$

- **POLAR**: simpler to subtract exponents

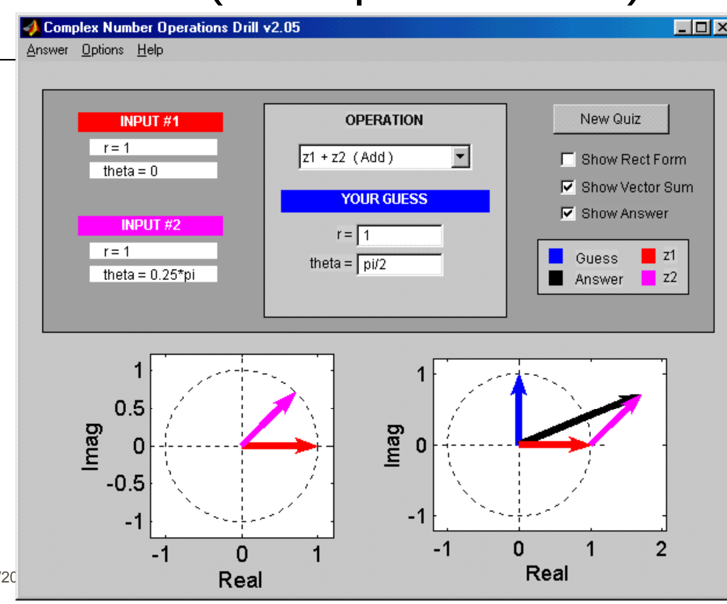
$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

COMPLEX MULTIPLY = VECTOR ROTATION

- Multiplication/division scales and rotates vectors

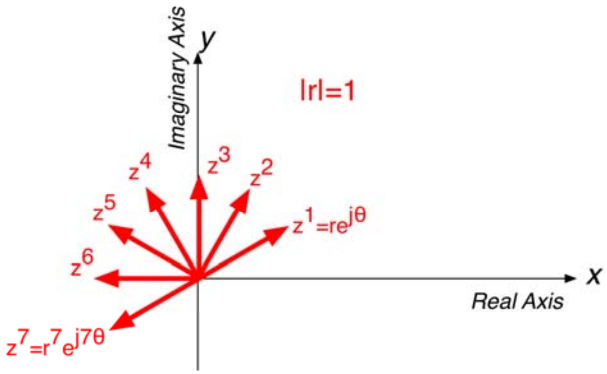


Z DRILL (Complex Arith)

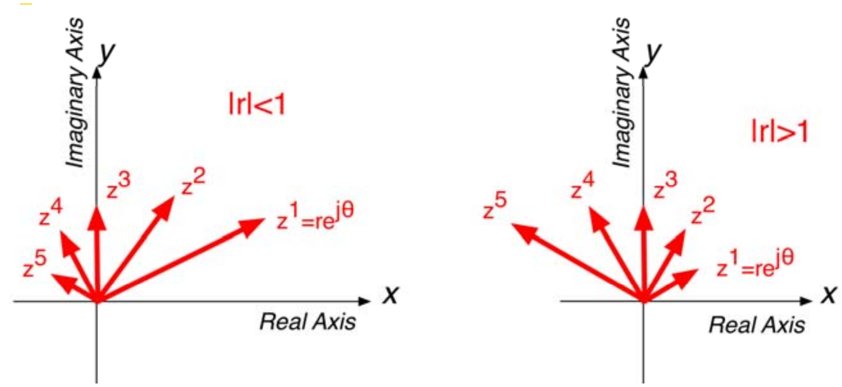


POWERS

- Raising to a power N rotates vector by Nθ and scales vector length by r^N



MORE POWERS



ROOTS OF UNITY

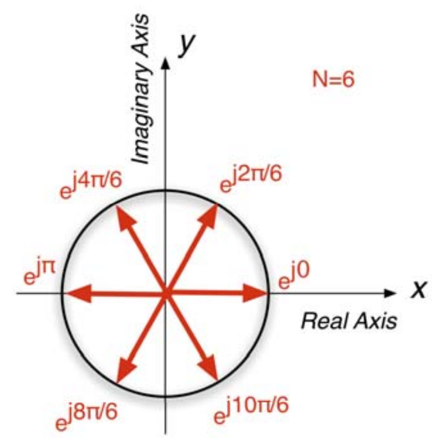
- We often have to solve $z^N=1$
- How many solutions?

$$z^N = r^N e^{jN\theta} = 1 = e^{j2\pi k}$$

$$\Rightarrow r = 1, \quad N\theta = 2\pi k \Rightarrow \theta = \frac{2\pi k}{N}$$

$$z = e^{j2\pi k/N}, \; k = 0,1,2,\dots N-1$$

ROOTS OF UNITY for N=6



- Solutions to $z^N=1$ are N equally spaced vectors on the unit circle!

Sum the Roots of Unity

- Looks like the answer is zero (for N=6)

$$\sum_{k=0}^{N-1} e^{j2\pi k/N} = 0?$$

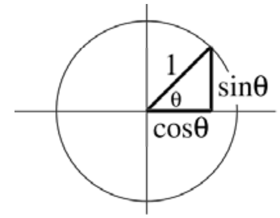
$$\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} \quad \text{then let } r = e^{j2\pi/N}$$

$$\text{Numerator} \quad 1-r^N = 1-(e^{j2\pi/N})^N = 1-e^{j2\pi} = 0$$

COMPLEX EXPONENTIAL

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
  - **θ = ω*t***
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs



$$\theta = \omega t$$

Integrate Complex Exp

- Needed in Fourier Series
  - Especially over one period

$$\int_a^b e^{j\theta} d\theta = \left.\frac{e^{j\theta}}{j}\right|_a^b = \frac{e^{jb} - e^{ja}}{j}$$

$$\int_0^T e^{j2\pi t/T} dt = \frac{e^{j2\pi T/T} - e^{j0}}{j} = \frac{1-1}{j} = 0$$

BOTTOM LINE

- **CARTESIAN**: Addition/subtraction is most efficient in Cartesian form
- **POLAR**: good for multiplication/division
- STEPS:
  - Identify arithmetic operation
  - Convert to easy form
  - Calculate
  - Convert back to original form