Learning and Pricing with Models that Do Not Explicitly Incorporate Competition

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Abstract

In revenue management research and practice, demand models are used that describe how demand for a seller's products depends on the decisions, such as prices, of that seller. Even in settings where the demand for a seller's products also depends on decisions of other sellers, the models often do not explicitly account for such decisions. It has been conjectured in the revenue management literature that such monopoly models may incorporate the effects of competition, because the parameter estimates of the monopoly models are based on data collected in the presence of competition. In this paper we take a closer look at such a setting to investigate the behavior of parameter estimates and decisions if monopoly models are used in the presence of competition. We consider repeated pricing games in which two competing sellers use mathematical models to choose the prices of their products. Over the sequence of games, each seller attempts to estimate the values of the parameters of a demand model that expresses demand as a function only of its own price using data comprised only of its own past prices and demand realizations. We analyze the behavior of the sellers' parameter estimates and prices under various assumptions regarding the sellers' knowledge and estimation procedures, and identify situations in which (a) the sellers' prices converge to the Nash equilibrium associated with knowledge of the correct demand model, (b) the sellers' prices converge to the cooperative solution, and (c) the sellers' prices have many potential limit points that are neither the Nash equilibrium nor the cooperative solution and that depend on the initial conditions. We compare the sellers' revenues at potential limit prices with their revenues at the Nash equilibrium and the cooperative solution, and show that it is possible for sellers to be better off when using a monopoly model than at the Nash equilibrium.

1 Introduction

Revenue management (RM) models are used by many businesses to make operational pricing and availability decisions. Even in settings with multiple competing sellers, each seller typically uses a model as though the seller is a monopolist, and thus these models do not explicitly account for the effects of competitors' decisions. Sellers may consider competitive effects at a strategic level or may sometimes account for competition in an ad hoc manner when making operational decisions, but in many revenue management systems competition is not explicitly modeled at the operational level. Phillips (2005, p. 59) states "There does not appear to be a single pricing and revenue optimization system that explicitly attempts to forecast competitive response using game theory as part of its ongoing operation." Nevertheless, competitors' decisions typically do interact with a seller's own decisions to affect sales and revenues. A seller may attempt to develop a demand model that takes the effect of some competitors into account, but it is likely that there would remain competitors whose effects are still not included in the model. For example, an airline may develop a model that takes into account the effect of a competing airline's prices on demand for its own tickets, but even then the model may not include the effects of train ticket or rental car prices.

In revenue management practice, settings that involve competing sellers often have the following elements:

[A] Each seller uses a model of demand or expected revenue as a function of its own decisions (e.g., prices or booking limits). The model is incorrect in the sense that it does not explicitly incorporate the effect of competitors' decisions on demand or revenue.

[B] Each seller uses data comprised *only* of its own past decisions and its own past demands (or, more accurately, sales) to estimate the parameters of its model.

[C] With the parameter estimates in hand, each seller then treats its model and the associated parameter estimates as if they were correct and optimizes the objective of the model (usually expected profit or expected revenue) to make a decision.

[D] As new data are obtained, each seller updates its parameter estimates with the hope of getting better estimates and making better decisions.

Section 5.1.4.3 of Talluri and van Ryzin (2004) contains some comments on the ubiquity of monopoly models in RM practice, and their Chapter 9 discusses RM forecasting. However, there seems to be little, if any, discussion of the dynamics resulting from [A]–[D] in the RM literature.

Given the widespread use of monopoly models, it is not surprising that much of the technical literature on RM does not consider competition. There are some papers that consider competitive settings in which sellers accurately model themselves and their competition, and that focus on identifying pricing or inventory policies that constitute Nash equilibria. However, as mentioned above, the use of models that explicitly include competition is not typical in RM practice. Also, most work, with and without a model of competition, does not consider the possibility that the decision maker's model is incorrect; that is, the possibility that there may not exist parameter values such that the resulting model constitutes an accurate representation of the decision problem for which it is intended to be used. For example, it is often assumed that each time period's demand is a random variable with distribution that depends only on the current price of the seller, and not on previous prices of the seller or on the prices of other sellers. However, in many applications it is likely that demand depends on previous prices, for example, buyers may use previous prices to forecast future prices, or buyers may exhibit behavioral traits such as the reference price effect. Also, in many applications it is likely that the demand for a seller's product depends on the prices of other sellers. In spite of such modeling error being obviously possible and even being likely, most RM research and applied work do not consider the possibility that the decision maker's model may be incorrect.

There is some work that considers models with unknown parameter values, e.g., unknown demand distributions, but still the basic assumption is that there exist values of the parameters that make the model an accurate representation of the decision problem. Much of the literature assumes in addition that the correct parameter values are known by the decision maker. For example, the probability distribution of demand is often assumed to be known for all potential price settings. Hence, most revenue management literature can be viewed as focusing exclusively on element [C] above. (We provide a literature review in Section 3.)

It is sometimes conjectured that, although revenue management models usually do not explicitly incorporate competition, they possibly implicitly incorporate competition through parameter estimates that serve as inputs to the models. Specifically, the historical demand data used by a seller to estimate demand for its own product as a function of its own decisions are influenced by historical values of *both* its own price and competitors' prices. Therefore, the conjecture is that demand forecasts may gradually incorporate the effects of competition as more data are collected, and the models that use those forecasts may consequently take implicit account of competition. Talluri and van Ryzin (2004, p. 186) attribute the following idea to "conventional wisdom": "[A]n observed historical price response has embedded in it the effects of competitors' responses to the firm's pricing strategy. So for instance, if a firm decides to lower its price, the firm's competitors might respond by lowering their prices. With market prices lower, the firm and its competitors see an increase in demand. The observed increase in demand is then measured empirically and treated as the 'monopoly' demand response to the firm's price change in a dynamic-pricing model — even though competitive effects are at work." The idea is also raised by Phillips (2005, p. 55), who notes in a discussion of pricing and competition that historical effects of competition are built into an individual seller's estimate of its price-response function (the function that specifies its demand as a function of just its own price). We call the conjecture that the effects of competition are implicitly captured in monopoly models estimated with observed data the *Market Response Hypothesis.* To our knowledge, there has been no work in the RM literature that attempts to study to what extent the Market Response Hypothesis is correct. Talluri and van Ryzin (2004, p. 186) also issue a warning that the use of monopoly models calibrated with data that have embedded the effects of competitors' responses runs the risk of reinforcing "bad" equilibrium responses.

In this paper we will be particularly interested in understanding the long-run behavior of revenue management systems in the presence of elements [A]–[D]. Do parameter estimates and prices converge as more data is accumulated, and if so, to what? To what extent do estimated monopoly models implicitly incorporate competition? Are sellers whose models take competition into account better off than sellers whose models do not?

Our main results establish that prices and parameter estimates converge in some settings, that the limit may correspond to the Nash equilibrium, or the cooperative solution at which the combined revenue of the sellers is maximized, or another outcome, and that in some settings the long-run behavior is unpredictable in the sense that there are many potential limits. The analysis reveals that in some settings monopoly models implicitly incorporate competition through parameter estimates in a rather limited fashion, and in other settings not at all. Although the monopoly models generate correct expressions for expected demand at the limit prices, in many cases they provide incorrect values for expected demand away from the limit prices. Thus, one should not rely on the Market Response Hypothesis. Notwithstanding the warning cited above, we also find that in many cases, the sellers are better off if they use incorrect models that do not explicitly incorporate competition than they would be if they knew the precise expected demand as a function of prices and followed the standard solution prescribed by the Nash equilibrium. In fact, when each is using an incorrect model, it is possible for the sellers to unwittingly end up at the cooperative solution.

The remainder of this paper is organized as follows. In Section 2 we introduce the framework that will be used throughout the paper and provide an overview of our main results. In Section 3 we review the literature. Section 4 contains our main results. We present concluding remarks in Section 5. Proofs that are not presented in the text can be found in the Appendix.

2 Preliminaries and Overview of Results

Consider a duopoly with two sellers, called seller 1 and seller -1. Each seller sells a product, and the product of seller i will be called product i. Suppose each seller i chooses a price p_i for product i, and, in response, quantity d_i of product i is requested by buyers from seller i. In this paper we focus on the case of linear demand, that is,

$$d_i \quad = \quad d_i(p_i, p_{-i}) + \varepsilon_i \qquad i = \pm 1 \tag{1}$$

where

$$d_i(p_i, p_{-i}) \quad = \quad \beta_{i,0} + \beta_{i,i} p_i + \beta_{i,-i} p_{-i} \qquad i = \pm 1 \tag{2}$$

is the expected demand for product i and ε_i is a random variable with mean zero. Throughout, we assume that $\beta_{i,0} > 0$, $\beta_{i,i} < 0$ and $\beta_{i,-i} \geq 0$ for $i = \pm 1$ and that

$$\beta_{1,1}\beta_{-1,-1} \quad > \quad \beta_{1,-1}\beta_{-1,1} \ . \tag{3}$$

This condition means that each seller's own price has greater effect on its demand than does the other seller's price. The expected revenue of seller i is

$$g_i(p_i, p_{-i}) \quad := \quad p_i d_i(p_i, p_{-i}) \ . \tag{4}$$

Note that the sellers do not face any inventory or capacity constraints. Hence, the setting differs from that of revenue management problems in which sellers typically have a fixed amount of initial inventory to be sold. The prices that result from the sellers' learning and use of incorrect models will be compared with two benchmark pairs of prices, namely the Nash equilibrium pair of prices and the total revenue maximizing cooperative pair of prices. These two benchmarks are given next, before we introduce the sellers' models.

The Competitive Solution. When the two sellers do not collude, a typical solution concept for the problem is a Nash equilibrium (NE). In a (pure strategy) NE each seller chooses a price that is a best response to its competitor's price. The best response of seller i when its competitor sets price p_{-i}, is given by

$$p_i \quad = \quad \arg\max_{p_i} g_i(p_i, p_{-i}) \quad = \quad -\frac{\beta_{i,0} + \beta_{i,-i} p_{-i}}{2\beta_{i,i}} \tag{5}$$

Solving (5) simultaneously for $i = \pm 1$, it follows that the unique NE prices $(p_{-1}^{\mathrm{N}}, p_1^{\mathrm{N}})$ are given by

$$p_i^{\mathrm{N}} \quad = \quad \frac{\beta_{-i,0}\beta_{i,-i} - 2\beta_{i,0}\beta_{-i,-i}}{4\beta_{-i,-i}\beta_{i,i} - \beta_{-i,i}\beta_{i,-i}} \ . \tag{6}$$

The Cooperative Solution. Also of interest is the cooperative solution in which the sellers collude to set prices that maximize the expected total revenue,

$$\bar{g}(p_{-1}, p_1) \quad := \quad g_{-1}(p_{-1}, p_1) + g_1(p_1, p_{-1}) \ . \tag{7}$$

When $4\beta_{-1,-1}\beta_{1,1} - (\beta_{-1,1} + \beta_{1,-1})^2 > 0$, the first order conditions are necessary and sufficient to maximize (7), and it follows that the resulting prices are

$$p_i^{\mathrm{C}} \quad = \quad \frac{\beta_{-i,0}\left(\beta_{-i,i} + \beta_{i,-i}\right) - 2\beta_{i,0}\beta_{-i,-i}}{4\beta_{-i,-i}\beta_{i,i} - \left(\beta_{-i,i} + \beta_{i,-i}\right)^2} \ . \tag{8}$$

As a simple check, it is straightforward to verify that $\bar{g}(p_{-1}^{\mathrm{C}}, p_1^{\mathrm{C}}) \geq \bar{g}(p_{-1}^{\mathrm{N}}, p_1^{\mathrm{N}})$.

Modeling Error. The focus of this paper is the use of models that do not explicitly incorporate competition. Specifically, we consider a situation in which each seller i uses the following monopoly model for expected demand as a function of its own price only:

$$\delta_i(p_i) \quad = \quad \alpha_{i,0} + \alpha_i p_i \ . \tag{9}$$

Note that the model (9) of demand used by each seller is different from the actual demand function (2).

Modeling Error with Learning. Suppose that over a sequence of time periods the sellers face repeated instances of the pricing game described above. The games (equivalently, time periods) are indexed by k. In period $k+1$, each seller i chooses a price p_i^k for $k = 0, 1, 2, \ldots$. These prices yield demands $(d_{-1}^{k+1}, d_1^{k+1})$, which are given as in (1) by

$$d_i^{k+1} \quad := \quad d_i(p_i^k, p_{-i}^k) + \varepsilon_i^{k+1} \ , \tag{10}$$

where $d_i(\cdot)$ is given by (2), $\{(\varepsilon_{-1}^k, \varepsilon_1^k) \ : \ k = 1, 2, \ldots\}$ is a martingale difference noise ($\mathbb{E}[\varepsilon_i^{k+1} | \mathcal{F}^k] = 0$ with probability 1 (w.p.1) for all k), and there is an M such that $\mathbb{E}[(\varepsilon_i^{k+1})^2 | \mathcal{F}^k] \leq M$ w.p.1 for all k. Here, $\mathcal{F}^k$ denotes the history generated by $(\varepsilon_{-1}^1, \varepsilon_1^1), \ldots, (\varepsilon_{-1}^k, \varepsilon_1^k)$ and (p_{-1}^0, p_1^0). In revenue management, such repeated decisions are typical; for example, airlines repeatedly offer flights from the same origin to the same destination on the same day of the week and at the same time of day.

We consider the situation in which each seller i uses a monopoly model for expected demand as a function of its own price only. Notwithstanding (10), each seller's choice of p_i^k is guided by its use of the monopoly model (9). Of course each seller i must somehow estimate values for the parameters $\alpha_{i,0}$ and α_i. Each seller i generates estimates $\widehat{\alpha}_{i,0}^k$ and $\widehat{\alpha}_i^k$ of $\alpha_{i,0}$ and α_i for period $k+1$ based on the data the seller has accumulated up to that point: its *own* historical price-demand data $(p_i^0, d_i^1), (p_i^1, d_i^2), \ldots, (p_i^{k-1}, d_i^k)$. The estimates $\widehat{\alpha}_{i,0}^k$ and $\widehat{\alpha}_i^k$ yield an estimated demand function, $\widehat{\delta}_i^k(p_i) := \widehat{\alpha}_{i,0}^k + \widehat{\alpha}_i^k p_i$. The construction of the estimates $\widehat{\alpha}_{i,0}^k$ and $\widehat{\alpha}_i^k$ from $(p_i^0, d_i^1), (p_i^1, d_i^2), \ldots, (p_i^{k-1}, d_i^k)$ is described later. We will mainly be interested in estimators that are such that if the chosen prices converge, then the estimated expected demand (given by the estimated demand function) at the chosen prices also converges to the actual expected demand at the limit prices. We term this the demand consistency property. We will provide a precise definition of the property later.

Once the estimates are determined, then each seller i chooses the price that maximizes its estimated expected revenue; i.e., seller i solves $\max_{p_i} \left\{p_i \widehat{\delta}_i^k(p_i)\right\} = \max_{p_i} \left\{p_i[\widehat{\alpha}_{i,0}^k + \widehat{\alpha}_i^k p_i]\right\}$, which yields price

$$p_i^k \quad = \quad -\frac{\widehat{\alpha}_{i,0}^k}{2\widehat{\alpha}_i^k} \tag{11}$$

provided that $\widehat{\alpha}_i^k < 0$. The general approach wherein decisions (here, the prices) are determined by an optimization that treats parameter estimates as if they were correct is sometimes called certainty equivalent control.

Market Response Hypothesis. The price of seller i affects the demand of seller $-i$, and therefore the prices of seller i may eventually affect the prices selected by seller $-i$, even when seller $-i$ does not observe the prices of seller i. Therefore, different prices for seller i may result in different prices for seller $-i$. To represent this possibility that the price chosen by a seller may be different depending on the price chosen by the other seller, suppose that the reaction of seller $-i$ to the pricing decision of the other seller i can be represented with a reaction function; if seller i sets price p_i, then seller $-i$ will set price $p_{-i}(p_i)$.

Next we give a few remarks and an example, and then we give a precise statement of the Market Response Hypothesis. First, existence of a reaction function $p_{-i}(p_i)$ does not imply that seller $-i$

consciously tracks the pricing decisions of seller i and devises a response to those decisions. In fact, reaction functions arise implicitly when the sellers use the learning approaches discussed in this paper to estimate the parameters of their monopoly models (9), and use (11) to set their prices. Second, a number of different reaction functions may be of interest depending on the amount of data collected by seller $-i$ both before seller i sets the price at p_i and while seller i keeps the price at p_i. For example, suppose that seller $-i$ uses the monopoly model (9), and after many demand observations has settled on a price p°_{-i}. If seller i sets a price p_i (that may or may not be different from previous prices) over the next few time periods, then the price of seller $-i$ is likely to remain constant at p°_{-i}. In this case we may take the reaction function to be $p_{-i}(p_i) = p^\circ_{-i}$. We consider this and another reaction function later.

Given a reaction function $p_{-i}(p_i)$, we say that the monopoly model of seller i *captures competition* if its parameters $\alpha_{i,0}$ and α_i in (9) are set so that

$$\delta_i(p_i) = d_i(p_i, p_{-i}(p_i)) \quad \text{for all } p_i. \tag{12}$$

We can now state the Market Response Hypothesis as follows. When sellers estimate the parameters of monopoly models with data collected under competition, their parameter estimates will converge to a limit at which each seller's monopoly model captures competition.

Here we do some preliminary analysis to understand under what conditions the Market Response Hypothesis is plausible. Consider seller i. For its monopoly model to capture competition, the reaction function $p_{-i}(p_i)$ has to be affine, say $p_{-i}(p_i) = \gamma_0 + \gamma_1 p_i$. In that case, (12) holds if and only if it happens to be that $\alpha_{i,0} = \beta_{i,0} + \beta_{i,-i}\gamma_0$ and $\alpha_i = \beta_{i,i} + \beta_{i,-i}\gamma_1$. Thus, whether the Market Response Hypothesis holds depends on the sellers' parameter estimates $\alpha_{i,0}$ and α_i of (9), as well as their reaction functions (in addition to the true parameter values). Regarding the sellers' parameter estimates, in the rest of the paper we investigate the sellers' estimation of their parameters and the limits of their estimates in various settings. Regarding the sellers' reaction functions, we will consider short-run and long-run price reaction functions (to be introduced later), both of which are affine and consistent with (11), and we will check whether they satisfy the Market Response Hypothesis in combination with the limits of the sellers' parameter estimates.

Overview of Main Results. In the remainder of the paper, we study the evolution of the prices (11) under different assumptions on the estimators $\widehat{\alpha}^k_{i,0}$ and $\widehat{\alpha}^k_i$. The goal is to understand the behavior of revenue management systems under elements [A]–[D] described in the previous section. Our main results describe the long-run behavior of parameter estimates and prices when the sellers try to learn about the incorrect model (9). We consider three separate settings, which are studied in Sections 4.1, 4.2, and 4.3. Here, we provide a summary of the three settings.

In the first setting, each seller i believes that α_i in (9) is equal to $\beta_{i,i}$, and tries to learn $\alpha_{i,0}$. This represents a situation in which each seller understands how its own price affects its own demand, but does not directly account for how its competitor's price does. In this setting, we prove that the estimates of $\alpha_{i,0}$ converge almost surely, from which it follows that the prices converge almost surely as well. We also show that the limit of the prices is the Nash equilibrium associated with knowledge of the correct demand model. Based on such a convergence result, one might hope that in other settings reasonable learning methods will also lead to the Nash equilibrium, even if the sellers' models are incorrect.

However, this is not the case in our second setting, in which each seller i believes that $\alpha_{i,0}$ in (9) is equal to $\beta_{i,0}$, and tries to learn α_i. This represents a situation in which each seller knows the total market size (in the sense of knowing the intercept of the demand function), and tries to learn how its own price affects demand, while failing to directly account for the effect of its competitor's prices. In this setting, the analysis is considerably more complicated than in the first, and we limit our convergence analysis to "symmetric" sellers for which $d_{-1}(p_{-1}, p_1) = d_1(p_1, p_{-1})$ in (2). (Such symmetric sellers will still generally have different parameter estimates and prices.) We again find that the parameter estimates and prices converge almost surely. However, in contrast to the first setting, the limit prices are not a Nash equilibrium. Rather, the limit prices turn out to be equal to the cooperative solution. Although we do not have convergence results when parameters are not symmetric, it is possible to identify *necessary* conditions for a pair of prices to be limit prices. Using these necessary conditions, we can deduce that it is not generally the case that there will be convergence to the cooperative solution when parameters are asymmetric. Nevertheless, we find surprisingly that the sellers' revenues at these potential limit prices are never Pareto inferior to the NE so long as $\beta_{i,-i} \neq 0$.

In the third setting, we consider asymmetric sellers, each of which uses least squares to estimate both parameters in (9). Each seller i produces estimates $\widehat{\alpha}_{i,0}^k$ and $\widehat{\alpha}_i^k$ of $\alpha_{i,0}$ and α_i using least squares with the data $(p_i^0, d_i^1), \dots, (p_i^{k-1}, d_i^k)$. In this case we consider deterministic systems, i.e., $\varepsilon_i^k = 0$ in (10), and show that prices converge (and, in fact, reach the limit in the fourth period and thereafter remain there) if the initial prices satisfy certain conditions. Interestingly, the limit prices depend on the initial conditions. It is notable that among the attainable limits are price pairs that are Pareto superior and Pareto inferior to the Nash equilibrium. Moreover, both the

Nash equilibrium and the cooperative solution can be obtained as such limits.

It turns out that the Market Response Hypothesis holds in some special cases, but (12) does not hold in general, that is, monopoly models calibrated with data observed under competition do not, in general, capture competition in the limit. The monopoly models in our study do incorporate competition in some more limited ways. For instance, we will see that certain “equilibrium” prices arise in the limit from the parameter estimation process. The monopoly models with the limits of the estimates in place provide accurate values of the expected demand at those equilibrium prices (but not at other prices).

It is important to put our results in perspective in relation to the existing literature. Our models can be viewed as duopolies where each seller has an incorrect demand model and uses certainty equivalent control to choose the price in each period. As we will discuss below, even the setting of a true monopolist that knows the correct structure of the demand model and must estimate its parameters is an active topic of research. There are fewer results for the analogous duopoly problem in which both sellers know the correct structure of the demand model and combine parameter estimation and reasonable price adjustment schemes. In an earlier version of this paper we considered such a setting and showed that if both sellers use Cournot adjustment and certainty equivalent control to choose prices in each time period, and if parameter estimates converge (not necessarily to the correct values), then the chosen prices converge to a Nash equilibrium with respect to the limit parameter estimates. However, even in the monopoly setting, parameter estimates may not converge to the correct values if certainty equivalent control is used. Since the monopoly setting is a special case of the duopoly setting in which one seller does not adjust prices, convergence of parameter estimates to the correct values cannot be guaranteed in the duopoly setting in general if certainty equivalent control is used. In view of the difficulty of problems in which sellers know the correct structure of demand and try to estimate parameter values, it is not surprising that the duopoly setting in which sellers estimate parameters of incorrect models poses significant challenges.

3 Literature Review

Our work is closely related to a series of papers by Alan Kirman on modeling error in duopoly pricing. The papers consider repeated price competition between two sellers, each of which attempts to learn parameters of the demand model (9), which he calls the “perceived model”. Kirman uses the term “true model” to describe the actual relationship between prices and demands, and assumes that demands are deterministic linear functions of the prices of both sellers [$\varepsilon_i = 0$ in (1)].

In the first paper in the series, Kirman (1975) proves that if each of two symmetric sellers knows the true slope of its own deterministic demand as a function of its own price, and estimates only the intercept of its perceived model, then the sequence of price pairs chosen by the sellers converges to the Nash equilibrium of the pricing game in which both sellers know the true model. Interestingly, the dynamics of the price pairs turn out to be identical to those that arise when each seller knows the true demand model (including the true parameter values) and chooses prices according to fictitious play. Our Proposition 1 in Section 4.1 generalizes the convergence result of Kirman (1975) to settings with asymmetric sellers that face random demand. The 1975 paper also briefly discusses a situation in which each seller uses least squares with its own price-demand data to estimate both the intercept and the slope of its perceived model. Here, the paper introduces the notion of a "pseudo equilibrium" — combinations of prices and parameter estimates that generate demand data such that neither seller will change its prices or estimates. Finally, the paper suggests, based on a simulation study, that there is convergence to such a pseudo equilibrium, but that the associated limit prices depend on the initial prices.

Kirman (1983) builds on the 1975 paper, and focuses on the case in which each of two symmetric sellers uses least squares to estimate the intercept and slope of its perceived model. The main result of the 1983 paper is to identify those price pairs at which both sellers will remain from the fourth period onward, provided that the sellers set particular prices in the first three periods. The result establishes that for the particular initial prices, the sellers' prices do indeed converge and that the limit depends on the prices in the first three periods. When the initial three periods' prices are chosen as suggested, then a pseudo equilibrium prevails from the fourth period onward. The paper shows that although any pair of prices can be part of a pseudo equilibrium, only a certain subset of price pairs can be part of such a limit pseudo equilibrium. It also conjectures that the only possible limit prices are those in the identified subset, and cites the simulations of Kirman (1975) and Gates et al. (1977) as showing that prices converge from arbitrary initial conditions. It leaves open the issue of proving such convergence. Our Section 4.3 extends results of Kirman (1983) to settings with asymmetric sellers.

Brousseau and Kirman (1992) take up the issue of convergence, and argue that prices, in fact, do not converge when starting from *arbitrary* initial prices. They attribute the apparent convergence seen in the previous simulation studies to the declining weights placed on new observations by the sellers' estimation, which cause prices and parameter estimates to change very slowly over periods.

In addition, they argue that convergence occurs only when either (i) the first three periods' prices are chosen as in Kirman (1983), or (ii) the first two periods' prices are chosen so that each seller's estimated demand curve from the third period onward is perfect in the sense that all its price-demand pairs lie on the curve, including those from the first two periods. They refer to the latter case as a perfectly self-sustained equilibrium.

Kirman (1995) provides an overview of his earlier papers, and also surveys related work. Recent papers that consider the dynamics of duopoly or oligopoly price competitions and that focus on some variation of modeling error include Schinkel et al. (2002), Tuinstra (2004), and Chiarella and Szidarovszky (2005).

A related line of research in economics studies settings in which agents' expectations influence the dynamics of an economic system. A review of this work can be found in Evans and Honkapohja (2001). Much of this literature focuses on rational expectations equilibria, which can be characterized as fixed points of mappings that relate agents' perceptions of system dynamics and actual system dynamics. This literature does not appear to have considered the types of nonlinear equations that govern the dynamics of the sellers' prices and parameter estimates that we study.

Our study can be viewed in the broad context of learning in games; see Fudenberg and Levine (1998) for an overview. A considerable variety of learning strategies has been studied for games in which competitors have access to the true model (i.e., there are no issues of modeling error or parameter estimation) but do not know what the others will do. Well-known examples include Cournot adjustment and fictitious play. In the context of our pricing problem, if the sellers choose their prices using Cournot adjustment then the prices converge to the NE (a proof is available from the authors). Likewise, if the sellers select their prices using fictitious play, then the prices converge to the NE (see Kirman 1975 for a proof). Such results provide some justification for the use of Nash equilibrium as a solution concept. Even if the sellers do not know what the others will do and use simple rules to choose prices, the prices nevertheless converge to the NE. However, Cournot adjustment and fictitious play require that firms know the true functional form (2) of the expected demand as well as the specific values of the parameters. Even if sellers know the functional form (2), they typically have to learn about their demand. This would involve generating estimates of the parameters in (2) and updating the estimates as they accumulate data. It is an open question whether prices still converge to NE under reasonable parameter estimation and price adjustment methods, or whether the process can exhibit different behavior.

As indicated earlier, we are not aware of any research in the OR literature on RM that studies the effects of using models that do not take competition into account, in spite of the widespread use of such seemingly flawed models. The closest research to this paper is the work on the spiral-down effect by Cooper et al. (2006), who examine the long-run behavior of protection levels and forecasts that are generated by the use of models that do not accurately account for customer choice among fare classes. In a related paper, Lee et al. (2012) study newsvendor models in which the demand depends on the inventory quantities but the dependence is ignored by the decision maker. The paper analyzes the behavior of the dynamic optimization-estimation process when the empirical distribution is used to forecast demand. Cooper and Li (2012) study a setup similar to Cooper et al. (2006), but consider protection levels generated by a "buy-up" model. Besbes et al. (2010) consider a single seller who wants to choose a price to maximize expected revenue, but who does not know the true demand function. The seller restricts attention to a parameterized family of demand functions, calculates a parameter value that gives the best fit to observed data, and then chooses a price according to certainty equivalent control. A statistical procedure is proposed for testing whether the chosen price has true objective value sufficiently close to the true optimal objective value. The idea behind the approach is that the suitability of a model should be determined by the quality of decisions it produces, rather than the extent to which the model reflects reality. The approach allows possibly misspecified models. However, it does not consider the possibility that the objective value in a period may depend not only on the decision in the same period, but also on decisions in previous periods (e.g., because a competitor's decision in the current period depends on the seller's decisions in previous periods).

Several recent papers consider robust versions of revenue management problems. For example, Lim and Shanthikumar (2007) consider dynamic pricing problems in which the decision maker knows only that the true model is within some specified distance, as measured by relative entropy, from a nominal model. Lan et al. (2011) consider overbooking and allocation decisions with limited information using ideas from competitive analysis of algorithms, and focus on obtaining policies that yield the best worst-case performance. The paper also provides many additional references.

Other work in the OR literature considers revenue management problems with learning. For instance, van Ryzin and McGill (2000) and Kunnumkal and Topaloglu (2009) use stochastic approximation approaches to determine booking policies for revenue management problems, and prove that the obtained policies converge to an optimal policy. Such an approach does not require demand forecasting, but the convergence results essentially require knowledge of the correct form of the underlying objective function to correctly guide the direction of movement of the stochastic approximation iterates. These papers do not consider competition or modeling error.

Den Boer and Zwart (2014) consider a monopolist facing a linear demand function with unknown parameters, which it estimates using linear regression. They show that prices chosen according to certainty equivalent control can converge to a suboptimal value, and they propose what they term controlled variance pricing, which they prove to ensure almost sure convergence of the prices to the true optimal price. In addition, they provide bounds on the regret of the policy. Broder and Rusmevichientong (2012) also consider a monopolist facing a demand model with unknown parameters. They introduce an approach whereby the monopolist constructs maximum likelihood estimates of the unknown parameters and alternates between periods of implementing the certainty equivalent price and periods of implementing exploration prices. They provide bounds on the regret of the policy. Harrison et al. (2012) study a Bayesian setting in which a monopolist has a prior distribution on two possible demand models. They show that certainty equivalent control can lead to incomplete learning, wherein the monopolist never learns which of the two demand models is the correct one. If this occurs, then the monopolist's chosen prices converge to an uninformative — and suboptimal — price at which learning stops. They propose a method to avoid such negative outcomes by modifying the certainly equivalent prices slightly to keep them away from the uninformative price. Under the proposed method, they show that prices converge to the optimal price and they provide bounds on the regret. These papers do not consider competition or modeling error. Besbes and Zeevi (2009) consider a single instance of a dynamic pricing problem (rather than repeated instances of a static pricing problem considered in this paper and the three mentioned earlier in this paragraph) with an unknown demand model. They consider policies that begin with a learning phase followed by an exploitation phase. They show that the proposed policies are asymptotically optimal for large expected demands and large initial capacities. They compare parametric and non-parametric variations of their method, and point out that the comparative loss in performance that may come with implementing the non-parametric approach rather than the parametric approach can be viewed as the cost paid to avoid model misspecification.

Finally, there is also a body of work in the OR literature that focuses on game theoretic studies of revenue management and pricing problems. For example, Netessine and Shumsky (2005) consider a setting in which two airlines compete by choosing booking limits for discount tickets, and obtain conditions for the existence of a pure strategy Nash Equilibrium. Recent work that focuses on dynamic pricing with competition includes Perakis and Sood (2006), Levin et al. (2009), and Gallego and Hu (2014), among others. As evidenced by this growing literature, competition is an area of significant interest to the RM research community. Our work is distinguished from these papers because it considers the interactions of estimation, optimization, and modeling errors.

4 Dynamics of Some Misspecified Models

In this section we study three different settings in which each seller i uses the incorrect demand model $\delta_i(p_i) = \alpha_{i,0} + \alpha_i p_i$ given in (9) and attempts to learn the values of $\alpha_{i,0}$ (Section 4.1) or α_i (Section 4.2) or both (Section 4.3). In each of the three subsections, we describe the demand consistency property and the market response hypothesis as they pertain to the considered setting. We also present what we know about convergence of prices and parameter estimates, and we discuss the effects of modeling error on the sellers' revenues.

4.1 The Case with Known Slope

Consider a setting in which the sellers know (or believe) that the own-price coefficient α_i in (9) is equal to $\beta_{i,i}$ in (2), so that $\widehat{\alpha}_i^k = \beta_{i,i}$ for all k. Such a setting has been motivated in the economics literature as the result of sellers performing price experimentation in a neighborhood of current prices, see for example Silvestre (1977) and Tuinstra (2004). In neither of these papers nor here is such experimentation modeled. Each seller i constructs an estimator $\widehat{\alpha}_{i,0}^k$ with observed data $(p_i^0, d_i^1), \ldots, (p_i^{k-1}, d_i^k)$. Recall that each seller i chooses price p_i^k for period $k+1$ by maximizing its estimated profit function $p_i \widehat{\delta}_i^k(p_i) = p_i \left[\widehat{\alpha}_{i,0}^k + \beta_{i,i} p_i\right]$, and thus, consistent with (11), it follows that

$$p_i^k \quad = \quad -\frac{\widehat{\alpha}_{i,0}^k}{2\beta_{i,i}}. \tag{13}$$

Demand Consistency. We say that an estimator $\widehat{\alpha}_{i,0}^k$ has the *demand consistency property* if, whenever the prices (p_{-1}^k, p_1^k) converge to some limit $(p_{-1}^\infty, p_1^\infty) > 0$ as $k \to \infty$, then the estimated demand $\widehat{\delta}_i^k(p_i^k) = \widehat{\alpha}_{i,0}^k + \beta_{i,i} p_i^k$ converges to the expected demand at the limit prices, that is, $\widehat{\delta}_i^k(p_i^k) \to d_i(p_i^\infty, p_{-i}^\infty)$ (except perhaps on a set of probability zero). It follows that $\widehat{\alpha}_{i,0}^k \to \widehat{\alpha}_{i,0}^\infty := \beta_{i,0} + \beta_{i,-i} p_{-i}^\infty$, and $p_i^\infty = -\widehat{\alpha}_{i,0}^\infty/(2\beta_{i,i})$. Note that in the limit the demand predicted by the estimated model agrees with the actual expected demand $d_i(p_i^\infty, p_{-i}^\infty)$ at the limit prices, and thus the seller's estimated model appears correct to the seller. Also, the seller perceives that an optimal price is being chosen.

Next we find the set of all potential limit points of a pair of estimators $(\widehat{\alpha}_{-1,0}^k, \widehat{\alpha}_{1,0}^k)$ with the demand consistency property. Note that if $(\widehat{\alpha}_{-1,0}^k, \widehat{\alpha}_{1,0}^k)$ converges to a limit, then it follows from (13) that (p_{-1}^k, p_1^k) also converges, so that the condition in the definition of the demand consistency property can be applied. It follows from $\widehat{\alpha}_{i,0}^{\infty} = \beta_{i,0} + \beta_{i,-i}p_{-i}^{\infty}$ and $p_i^{\infty} = -\widehat{\alpha}_{i,0}^{\infty}/(2\beta_{i,i})$ for $i = \pm 1$ that the only possible limits of $(\widehat{\alpha}_{-1,0}^k, \widehat{\alpha}_{1,0}^k)$ are

$$\widehat{\alpha}_{i,0}^{\infty} = \frac{2\beta_{i,i}\left(2\beta_{i,0}\beta_{-i,-i} - \beta_{-i,0}\beta_{i,-i}\right)}{4\beta_{-i,-i}\beta_{i,i} - \beta_{-i,i}\beta_{i,-i}} \tag{14}$$

with corresponding limit prices

$$p_i^{\infty} = \frac{\beta_{-i,0}\beta_{i,-i} - 2\beta_{i,0}\beta_{-i,-i}}{4\beta_{-i,-i}\beta_{i,i} - \beta_{-i,i}\beta_{i,-i}}\,. \tag{15}$$

The above expression for the limit prices coincides with the Nash equilibrium associated with knowledge of the correct demand model given in (6). Also note that $\widehat{\alpha}_{i,0}^{\infty} = \beta_{i,0} + \beta_{i,-i}p_{-i}^{\infty}$ means that the effect of the competitor's price on the demand is swept into the estimated intercept parameter.

Based on the demand model (9) of seller i, the seller's observed intercept in time period k is $\widetilde{\alpha}_{i,0}^k := d_i^k - \beta_{i,i}p_i^{k-1}$. Consider any estimator $\widehat{\alpha}_{i,0}^k$ with the (reasonable to the seller) property that if the seller's expected observed intercept $d_i(p_i^{k-1}, p_{-i}^{k-1}) - \beta_{i,i}p_i^{k-1}$ converges to a limit, then the intercept estimator $\widehat{\alpha}_{i,0}^k$ converges to the same limit. It is easy to verify that any such estimator has the demand consistency property, and thus many estimators that would appear reasonable to the seller have the demand consistency property.

Market Response Hypothesis. It is evident from (12) that whether or not the Market Response Hypothesis holds depends on the reaction function $p_{-i}(p_i)$. Below, we consider two different reaction functions, both motivated by the following statement of the Market Response Hypothesis from Phillips (2005, p. 55): "To the extent that competitors will behave in a similar fashion in the future as they have in the past, the price-response function will be a fair representation of market response — including competitive response." We refer to the two reaction functions as the short-run reaction function and the long-run reaction function. For both, we imagine that parameter estimates and prices have converged and that seller i subsequently changes its price away from its limit price p_i^{∞}, and we ask what will happen to the price of seller $-i$ provided that seller $-i$ continues to "behave in a similar fashion in the future as (it has) in the past." We emphasize that this does not mean that seller i actually changes its price to some $p_i \neq p_i^{\infty}$ or that it has an incentive to do so. Rather, for the purpose of identifying a reasonable reaction function for seller $-i$ and evaluating the Market Response Hypothesis, we are merely asking the question what would happen *if* seller i were to make such a change to price p_i.

If the sellers' parameter estimates converge (this assumption gives the Market Response Hypothesis some "benefit of the doubt" because convergence of the estimates is part of the hypothesis), then the limits of the estimates and prices are given by (14) and (15). When the limit has been "reached", seller $-i$ sets price p_{-i}^{∞}. Moreover, a hypothetical change of the price of seller i from p_i^{∞} to some $p_i \neq p_i^{\infty}$ would, in the short run after the limit has been reached, not cause much movement in the parameter estimates of player $-i$. Hence, in the short run, the price of seller $-i$ would remain essentially fixed at p_{-i}^{∞}. This motivates the short-run reaction function of $p_{-i}(p_i) = p_{-i}^{\infty}$.

In the limit, seller i has an estimated demand function given by $\widehat{\delta}_i^{\infty}(p_i) := \widehat{\alpha}_{i,0}^{\infty} + \beta_{i,i} p_i$. It follows from (14) and (15) that

$$\widehat{\delta}_i^{\infty}(p_i) \quad = \quad \beta_{i,0} + \beta_{i,-i} p_{-i}^{\infty} + \beta_{i,i} p_i \quad = \quad d_i(p_i, p_{-i}^{\infty}) \qquad \text{for all } p_i \, . \tag{16}$$

Therefore, seller i has a correct estimate of its mean demand as a function of its own price if the price of seller $-i$ is held at p_{-i}^{∞}. That is, in the limit, the monopoly model $\widehat{\delta}_i^{\infty}(p_i)$ of seller i captures competition for the short-run reaction function $p_{-i}(p_i) = p_{-i}^{\infty}$. Hence, for this particular reaction function, the Market Response Hypothesis holds, provided that parameter estimates do indeed converge.

If one is interested in a longer time horizon during which seller $-i$ will "behave in a similar fashion in the future as (it has) in the past" after the limit has been reached, then it is appropriate to consider a different, long-run reaction function. If the limit has been reached and thereafter seller i repeatedly implements a price p_i for many periods, then eventually the intercept estimate of seller $-i$ will move away from $\widehat{\alpha}_{-i,0}^{\infty}$. If the estimate re-converges, it will do so to $\bar{\alpha}_{-i,0}^{\infty} = \beta_{-i,0} + \beta_{-i,i} p_i$ by the demand consistency property. (To see this, observe that if the intercept estimate of seller $-i$ converges to some value $\bar{\alpha}_{-i,0}^{\infty}$ then that seller's price converges to $\bar{p}_{-i}^{\infty} = -\bar{\alpha}_{-i,0}^{\infty}/(2\beta_{-i,-i})$. The demand consistency property then tells us that

$$\bar{\alpha}_{-i,0}^{\infty} + \beta_{-i,-i} \bar{p}_{-i}^{\infty} = \beta_{-i,0} + \beta_{-i,-i} \bar{p}_{-i}^{\infty} + \beta_{-i,i} p_i$$

from which it follows that $\bar{\alpha}_{-i,0}^{\infty} = \beta_{-i,0} + \beta_{-i,i} p_i$.) Consequently, the price of seller $-i$ will converge to $-(\beta_{-i,0} + \beta_{-i,i} p_i)/(2\beta_{-i,-i})$ by (13). In this case, it is natural to take the long-run reaction function of seller $-i$ to be $p_{-i}(p_i) = -(\beta_{-i,0} + \beta_{-i,i} p_i)/(2\beta_{-i,-i})$. For the long-run reaction function, the monopoly model does not capture competition because $\widehat{\delta}_i^{\infty}(p_i) = \beta_{i,0} + \beta_{i,i} p_i + \beta_{i,-i} p_{-i}^{\infty}$ cannot be equal to $d_i(p_i, p_{-i}(p_i)) = \beta_{i,0} + \beta_{i,i} p_i - \beta_{i,-i}(\beta_{-i,0} + \beta_{-i,i} p_i)/(2\beta_{-i,-i})$ for all p_i except in the trivial case when $\beta_{-i,i} = 0$, and therefore the Market Response Hypothesis does not hold.

In conclusion, under competition, the monopoly demand model $\widehat{\delta}_i^{\infty}(p_i) := \widehat{\alpha}_{i,0}^{\infty} + \beta_{i,i} p_i$ with $\widehat{\alpha}_{i,0}^{\infty}$ given by (14) is a fair representation of market response in the short run after the limits have been reached, but does not represent long-run market response. A seller may believe its monopoly model has adequately incorporated the effects of competition, because "manual checks" through price experimentation to see if observed demand matches predicted demand will initially not show anything wrong. As time progresses, however, such a belief would turn out to be false.

Convergence. Consider the estimator that simply averages the observed intercepts:

$$\widehat{\alpha}_{i,0}^k \quad := \quad \frac{1}{k} \sum_{j=1}^{k} \widetilde{\alpha}_{i,0}^j \,. \tag{17}$$

It is easy to check that $\widehat{\alpha}_{i,0}^k$ is the seller's least squares estimator of the parameter $\alpha_{i,0}$ when $\widehat{\alpha}_i^k$ is fixed to $\beta_{i,i}$. It follows from a strong law of large numbers for martingales (Chow 1967) that, if the sellers' models were correct, then $\widehat{\alpha}_{i,0}^k$ in (17) would be a consistent estimator for $\alpha_{i,0}$. It also follows that the estimator $\widehat{\alpha}_{i,0}^k$ has the demand consistency property. Thus the estimation procedure is reasonable, given the seller's demand model. Kirman (1975) showed that p_i^k in (13) converges to p_i^{∞} in (15) if estimator $\widehat{\alpha}_{i,0}^k$ in (17) is used in the case of symmetric sellers ($\beta_{i,0} = \beta_0$, $\beta_{i,i} = \beta_e$, $\beta_{i,-i} = \beta_d$ for $i = \pm 1$) and deterministic demand.

Note that by (2), (10), (13), and (17), we have

$$\widehat{\alpha}_{i,0}^k \quad = \quad \left(1 - \frac{1}{k}\right) \widehat{\alpha}_{i,0}^{k-1} + \frac{1}{k} \left[\beta_{i,0} - \frac{\beta_{i,-i}}{2\beta_{-i,-i}} \widehat{\alpha}_{-i,0}^{k-1} + \varepsilon_i^k\right].$$

It may be that $\widehat{\alpha}_{i,0}^k < 0$ in which case p_i^k will be negative. This is clearly unrealistic, so we will assume that each seller projects its estimate onto the positive half-line. That is, rather than (17), the intercept estimates will be

$$\widehat{\alpha}_{i,0}^k \quad := \quad \pi\left(\left(1 - \frac{1}{k}\right) \widehat{\alpha}_{i,0}^{k-1} + \frac{1}{k} \left[\beta_{i,0} - \frac{\beta_{i,-i}}{2\beta_{-i,-i}} \widehat{\alpha}_{-i,0}^{k-1} + \varepsilon_i^k\right]\right). \tag{18}$$

where $\pi(x) = \max\{x, 0\}$ for $x \in \mathbb{R}$. Such post-processing from "reasonableness checks" on parameter estimates is common in revenue management applications. Also observe that these projections do not require any coordination between the sellers — each seller i simply replaces its own estimate $\widehat{\alpha}_{i,0}^k$ by $\max\{\widehat{\alpha}_{i,0}^k, 0\}$.

Next we look at the long-run behavior of the iterates in (18). Let $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be given by

$$F(x_{-1}, x_1) \quad := \quad \left(\beta_{-1,0} - \frac{\beta_{-1,1}}{2\beta_{1,1}} x_1,\ \beta_{1,0} - \frac{\beta_{1,-1}}{2\beta_{-1,-1}} x_{-1}\right) \quad = \quad \begin{pmatrix} \beta_{-1,0} \\ \beta_{1,0} \end{pmatrix} + M \begin{pmatrix} x_{-1} \\ x_1 \end{pmatrix},$$

where

$$M = \begin{pmatrix} 0 & -\frac{\beta_{-1,1}}{2\beta_{1,1}} \\ -\frac{\beta_{1,-1}}{2\beta_{-1,-1}} & 0 \end{pmatrix}.$$

Then from (18),

$$\left(\widehat{\alpha}^k_{-1,0}, \widehat{\alpha}^k_{1,0}\right) = \bar{\pi}\left(\left(1 - \frac{1}{k}\right)\left(\widehat{\alpha}^{k-1}_{-1,0}, \widehat{\alpha}^{k-1}_{1,0}\right) + \frac{1}{k}\left[F\left(\widehat{\alpha}^{k-1}_{-1,0}, \widehat{\alpha}^{k-1}_{1,0}\right) + \left(\varepsilon^k_{-1}, \varepsilon^k_1\right)\right]\right),$$

where $\bar{\pi}(x_{-1}, x_1) := (\pi(x_{-1}), \pi(x_1))$ for $(x_{-1}, x_1) \in \mathbb{R}^2$. Note that $(\widehat{\alpha}^\infty_{-1,0}, \widehat{\alpha}^\infty_{1,0})$ given by (14) is the unique fixed point of F, and that F is a contraction mapping if and only if the eigenvalues of the matrix M are less than 1 in absolute value. The latter condition holds if and only if $|\beta_{-1,1}\beta_{1,-1}| < 4\,|\beta_{1,1}\beta_{-1,-1}|$, which holds by (3). Therefore, the following result is an immediate consequence of Proposition A–1 in the Appendix.

Proposition 1. *Consider the sequence* $\{(\widehat{\alpha}^k_{-1,0}, \widehat{\alpha}^k_{1,0})\}$ *defined in* (18). *Then w.p.1,* $\lim_{k\to\infty} \widehat{\alpha}^k_{i,0} = \widehat{\alpha}^\infty_{i,0}$ *for each* $i = \pm 1$, *where* $\widehat{\alpha}^\infty_{i,0}$ *is given by* (14).

If we do not consider projection and instead consider (17), then the same convergence follows by a small modification of the proof of Proposition A–1. It follows immediately from Proposition 1 that the sellers' prices converge to the Nash equilibrium prices (15) and the sellers' revenues converge to those associated with the Nash equilibrium.

4.2 The Case with Known Intercept

In this section we study the case in which each seller knows (or believes) that the intercept coefficient $\alpha_{i,0}$ in (9) is equal to $\beta_{i,0}$ in (2), so that $\widehat{\alpha}^k_{i,0} = \beta_{i,0}$ for all k. Each seller i constructs an estimator $\widehat{\alpha}^k_i$ with observed data. That is, seller i estimates a linear demand model

$$\widehat{\delta}^k_i(p_i) = \beta_{i,0} + \widehat{\alpha}^k_i p_i\ . \tag{19}$$

Note that, based on the demand model of seller i, the seller's observed slope in time period k is $\widetilde{\alpha}^k_i := (d^k_i - \beta_{i,0})/p^{k-1}_i$. As before, seller i chooses the price p^k_i that maximizes the revenue function $p_i\widehat{\delta}^k_i(p_i)$, which is

$$p^k_i = -\frac{\beta_{i,0}}{2\widehat{\alpha}^k_i} \tag{20}$$

consistent with (11).

Demand Consistency. As before, we say that an estimator $\widehat{\alpha}_i^k$ has the *demand consistency property* if, whenever the prices (p_{-1}^k, p_1^k) converge to some limit $(p_{-1}^\infty, p_1^\infty) > 0$ as $k \to \infty$, then the estimated demand $\widehat{\delta}_i^k(p_i^k) = \beta_{i,0} + \widehat{\alpha}_i^k p_i^k$ converges to the expected demand at the limit prices, that is, $\widehat{\delta}_i^k(p_i^k) \to d_i(p_i^\infty, p_{-i}^\infty)$. Therefore, $\widehat{\alpha}_i^k \to \widehat{\alpha}_i^\infty := \beta_{i,i} + \beta_{i,-i} p_{-i}^\infty / p_i^\infty$, and $p_i^\infty = -\beta_{i,0}/(2\widehat{\alpha}_i^\infty)$.

It follows that, for any pair of estimators $(\widehat{\alpha}_{-1}^k, \widehat{\alpha}_1^k)$, both with the demand consistency property, the only possible limits are

$$\widehat{\alpha}_i^\infty = \beta_i^* := \frac{\beta_{i,0}\left(\beta_{-i,i}\beta_{i,-i} - \beta_{-i,-i}\beta_{i,i}\right)}{\beta_{-i,0}\beta_{i,-i} - \beta_{i,0}\beta_{-i,-i}} \tag{21}$$

with corresponding limit prices

$$p_i^\infty = -\frac{\beta_{i,0}}{2\widehat{\alpha}_i^\infty} = \frac{\beta_{-i,0}\beta_{i,-i} - \beta_{i,0}\beta_{-i,-i}}{2(\beta_{-i,-i}\beta_{i,i} - \beta_{-i,i}\beta_{i,-i})} . \tag{22}$$

Note that $\beta_i^* < 0$ and $p_i^\infty > 0$.

Market Response Hypothesis. Does the Market Response Hypothesis hold in the current setting in which each seller estimates the slope of a monopoly model? To address this question, we again consider short-run and long-run reaction functions. The short-run reaction function of seller $-i$ is $p_{-i}(p_i) = p_{-i}^\infty$ where p_{-i}^∞ is given by (22). The motivation for this reaction function is identical to that provided in Section 4.1 for the short-run reaction function discussed there.

If parameter estimates converge, then seller i has monopoly model $\widehat{\delta}_i^\infty(p_i) := \beta_{i,0} + \widehat{\alpha}_i^\infty p_i$ in the limit, where $\widehat{\alpha}_i^\infty$ is given by (21). When both sellers implement their limit prices, the actual expected demand for seller i is $d_i(p_i^\infty, p_{-i}^\infty) = \beta_{i,0} + \beta_{i,i}p_i^\infty + \beta_{i,-i}p_{-i}^\infty = \beta_{i,0}/2$ and the model of seller i predicts an expected demand of $\widehat{\delta}_i^\infty(p_i^\infty) = \beta_{i,0} + \widehat{\alpha}_i^\infty p_i^\infty = \beta_{i,0}/2$. However, in contrast to (16) where the slope is known and the intercept is estimated, in this case $\widehat{\delta}_i^\infty(p_i) \neq d_i(p_i, p_{-i}^\infty)$ for $p_i \neq p_i^\infty$ (when $\beta_{i,-i} \neq 0$). Thus, even if the price of seller $-i$ were to be fixed at $p_{-i}(p_i) = p_{-i}^\infty$, the estimate of expected demand provided by the model of seller i would be correct only when the price of seller i is p_i^∞, rather than at all prices as in Section 4.1. Hence, in the limit the monopoly model of seller i does not capture competition in this setting for the short-run reaction function, and therefore the Market Response Hypothesis does not hold.

We now turn to the long-run reaction function. If after reaching the limit, seller i repeatedly sets price p_i, then in the long run, the slope estimate of seller $-i$ will move away from $\widehat{\alpha}_{-i}^\infty = \beta_{-i}^*$, and if the estimate re-converges it will do so to $\beta_{-i,0}\beta_{-i,-i}/(\beta_{-i,0} + 2\beta_{-i,i}p_i)$ by the demand consistency property. Consequently, the price of seller $-i$ will converge to $-(\beta_{-i,0} + 2\beta_{-i,i}p_i)/(2\beta_{-i,-i})$, and thus we take the long-run reaction function of seller $-i$ to be $p_{-i}(p_i) =$

$-(\beta_{-i,0} + 2\beta_{-i,i}p_i)/(2\beta_{-i,-i})$. For the long-run reaction function, it holds that $d_i(p_i, p_{-i}(p_i)) = [\beta_{i,0} - \beta_{-i,0}\beta_{i,-i}/(2\beta_{-i,-i})] + [\beta_{i,i} - \beta_{-i,i}\beta_{i,-i}/\beta_{-i,-i}]p_i$, but the monopoly model gives $\widehat{\delta}_i^{\infty}(p_i) = [\beta_{i,0}] + [\beta_{i,0}\left(\beta_{-i,i}\beta_{i,-i} - \beta_{-i,-i}\beta_{i,i}\right)/(\beta_{-i,0}\beta_{i,-i} - \beta_{i,0}\beta_{-i,-i})]p_i$. Hence the monopoly model of seller i does not capture competition and the Market Response Hypothesis does not hold for the long-run reaction function.

In conclusion, the monopoly demand model $\widehat{\delta}_i^{\infty}(p_i) := \beta_{i,0} + \widehat{\alpha}_i^{\infty}p_i$ where $\widehat{\alpha}_i^{\infty}$ is given by (21) represents neither short-run nor long-run market response under competition.

Effects of Modeling Error on Revenues. Next we evaluate the long-run effects of modeling error on the sellers' revenues, by considering the revenues associated with the prices p_i^{∞} in (22) that are selected when the slope estimates are $\widehat{\alpha}_i^{\infty}$ in (21). We refer to the prices $(p_{-1}^{\infty}, p_1^{\infty})$ given by (22) as the modeling error equilibrium (MEE).

The expected revenue of seller i at the MEE is given by $g_i(p_i^{\infty}, p_{-i}^{\infty})$, where g_i is given by (4). Likewise, the expected revenue of seller i at the Nash equilibrium (NE) is given by $g_i(p_i^{\mathrm{N}}, p_{-i}^{\mathrm{N}})$. Let

$$\Delta_i \quad := \quad \frac{g_i(p_i^{\infty}, p_{-i}^{\infty}) - g_i(p_i^{\mathrm{N}}, p_{-i}^{\mathrm{N}})}{g_i(p_i^{\mathrm{N}}, p_{-i}^{\mathrm{N}})}$$

denote the relative increase in expected revenue for seller i at the MEE in comparison to the NE. If Δ_i is positive (negative), then seller i is better (worse) off at the MEE than at the NE.

If both cross-price sensitivity terms are zero (if $\beta_{i,-i} = 0$ for $i = \pm 1$), then the sellers do not affect each other. In this case, it is easy to see that the MEE and the NE coincide and hence $\Delta_i = 0$ for $i = \pm 1$. If $\beta_{-1,1} = 0$ and $\beta_{1,-1} > 0$, then $\Delta_{-1} = 0$ and calculations show that $g_1(p_1^{\infty}, p_{-1}^{\infty}) - g_1(p_1^{\mathrm{N}}, p_{-1}^{\mathrm{N}}) = (\beta_{-1,0}\beta_{1,-1})^2/(16\beta_{-1,-1}^2\beta_{1,1}) < 0$ so that $\Delta_1 < 0$. Likewise, if $\beta_{1,-1} = 0$ and $\beta_{-1,1} > 0$, then $\Delta_1 = 0$ and $\Delta_{-1} < 0$. Hence, if one of the cross-price sensitivity terms is zero and the other is not, then the NE is Pareto superior to the MEE. As we show below, this is the only situation with known intercepts where the NE is Pareto superior to the MEE.

For the following comparison of the expected revenue under the NE and the MEE, assume that $\beta_{i,-i} = -\theta\beta_{i,i}$ for $i = \pm 1$ for some $\theta \in (0, 1)$; hence,

$$d_i(p_i, p_{-i}) \quad = \quad \beta_{i,0} + \beta_{i,i}p_i - \theta\beta_{i,i}p_{-i} \qquad i = \pm 1. \tag{23}$$

Lemma A–1 in the appendix shows that this entails no loss of generality because we can always re-scale the units of one seller's price to arrive at demand functions with this form. Below, we show $\beta := (\beta_{-1,-1}, \beta_{1,1}) \in \mathcal{S} := \{(x, y) : x < 0, y < 0\}$ as an argument of Δ_i to indicate dependence on $\beta_{-1,-1}$ and $\beta_{1,1}$.

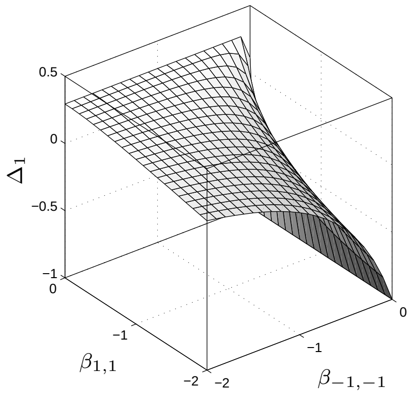

Figure 1: Relative revenue difference $\Delta_1(\beta)$.

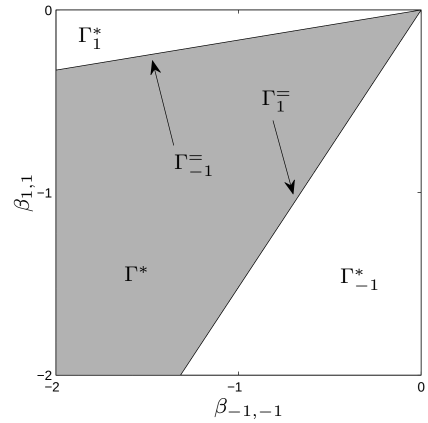

Figure 2: Regions of $\Delta_{-1}(\beta), \Delta_1(\beta)$.

Next we investigate the set of parameter values for which the sellers are better (or worse) off at the MEE than at the NE. Let $\Gamma_i := \{\beta \in \mathcal{S} : \Delta_i(\beta) \geq 0\}$. If $\beta \in \Gamma_i$, then seller i is better off at the MEE than at the NE. Likewise, let $\Gamma_i^{\mathrm{N}} := \{\beta \in \mathcal{S} : \Delta_i(\beta) \leq 0\}$ denote the region in which seller i does better at the NE than at the MEE. Finally, let $\Gamma_i^= := \{\beta \in \mathcal{S} : \Delta_i(\beta) = 0\}$ be the region where both the MEE and the NE yield the same revenue for seller i. Let $\Gamma^* := (\Gamma_{-1} \cap \Gamma_1) \setminus (\Gamma_{-1}^= \cap \Gamma_1^=)$ denote the set of parameter values β for which the MEE is Pareto superior to the NE, let $\Gamma_i^* := \Gamma_i \cap \Gamma_{-i}^{\mathrm{N}}$ denote the region where seller i is better off at the MEE, but seller $-i$ is better off at the NE, and let $\Gamma^{\mathrm{N}} := (\Gamma_{-1}^{\mathrm{N}} \cap \Gamma_1^{\mathrm{N}}) \setminus (\Gamma_{-1}^= \cap \Gamma_1^=)$ denote the region where the NE is Pareto superior to the MEE.

Figure 1 depicts $\Delta_1(\beta)$ for the case with $\beta_{-1,0} = 2$, $\beta_{1,0} = 1$, and $\theta = 0.6$. $\Delta_{-1}(\beta)$ looks similar to $\Delta_1(\beta)$, with the role of the axes for $\beta_{-1,-1}$ and $\beta_{1,1}$ interchanged. Figure 2 relates to the same example, and shows the set of $(\beta_{-1,-1}, \beta_{1,1})$ for which the MEE is Pareto superior to the NE. Note that there is a large set of $(\beta_{-1,-1}, \beta_{1,1})$ where $\Delta_{-1}(\beta), \Delta_1(\beta) > 0$. The figure also depicts the sets Γ_i^* for $i = \pm 1$. It is perhaps surprising to note that in the figure there are no $(\beta_{-1,-1}, \beta_{1,1})$-pairs at which the Nash equilibrium is Pareto superior to the modeling error equilibrium; that is, $\Gamma^{\mathrm{N}} = \varnothing$. The proposition below shows that, remarkably, this is always the case. Hence, in view of the proposition and Lemma A–1, the only setting with known intercepts where the NE is Pareto superior to the MEE is when one of the cross-price sensitivities is zero and the other is not.

In preparation for the next result, let

$$T(\theta) \quad := \quad \frac{(8+\theta^2)\theta + (4-\theta^2)\sqrt{8+\theta^2}}{8(1-\theta^2)} \qquad \text{for } \theta \in (0,1) \; .$$

Proposition 2. *Suppose that $d_i(p_i, p_{-i})$ is given by (23) for $\theta \in (0,1)$ for $i = \pm 1$. Given*

$\beta_{-1,0}, \beta_{1,0} > 0$, *let* $\rho_i := \beta_{i,0}/\beta_{-i,0} = 1/\rho_{-i}$ *for* $i = \pm 1$. *Then*

$$\begin{aligned}
\Gamma^* &= \{\beta \in \mathcal{S} \,:\, \rho_1 T(\theta)\beta_{-1,-1} \le \beta_{1,1} \le [\rho_1/T(\theta)]\beta_{-1,-1}\} \neq \varnothing \\
\Gamma_i^* &= \{\beta \in \mathcal{S} \,:\, \beta_{i,i} \ge [\rho_i/T(\theta)]\beta_{-i,-i}\} \neq \varnothing \qquad i = \pm 1 \\
\Gamma_i^= &= \{\beta \in \mathcal{S} \,:\, \beta_{i,i} = \rho_i T(\theta)\beta_{-i,-i}\} \neq \varnothing \qquad i = \pm 1 \\
\Gamma^{\mathrm{N}} &= \varnothing\,.
\end{aligned}$$

Proposition 2 establishes that, as suggested by Figure 2, the region Γ^* has linear boundaries. It can be verified that $T(\theta)$ is increasing; therefore, it follows that the region $\Gamma^* = \Gamma^*(\theta)$ is increasing in θ. (In this paragraph we show the dependence of Γ^* on θ.) Notably, $\Gamma^*(\theta)$ expands to all of $\mathcal{S}$ as $\theta \uparrow 1$. On the other hand, the region $\Gamma^*(\theta)$ shrinks to $\Gamma^*(0) := \{\beta \in \mathcal{S} \,:\, \sqrt{2}\rho_1\beta_{-1,-1} \le \beta_{1,1} \le [\rho_1/\sqrt{2}]\beta_{-1,-1}\}$ as $\theta \downarrow 0$. Note that the area of $\Gamma^*(0)$ is positive for all $\rho_1 > 0$. Consequently, there is a range of parameter values $\Gamma^*(0) \subset \mathcal{S}$ such that if $\beta \in \Gamma^*(0)$ then $\beta \in \Gamma^*(\theta)$ for all $\theta \in (0, 1)$. The proposition also shows that when $\beta_{-1,0}$ and $\beta_{1,0}$ are close to each other, and $\beta_{-1,-1}$ and $\beta_{1,1}$ are close to each other, then both sellers are better off under the MEE than under the NE.

It is natural to ask to what extent the Pareto superiority of MEE to NE depends on the assumption that the intercepts $\beta_{i,0}$ are known. Later we show that the Pareto superiority of MEE to NE can occur even in the more general case where the intercepts are unknown. However, in that case it also may be that the NE is Pareto superior to the MEE, depending on the initial conditions.

Convergence. Next we consider the question of convergence of parameter estimates and prices. Our goal is to show that the MEE is indeed achieved as the limit of the iterative estimation and pricing procedure.

Consider a slope estimator $\widehat{\alpha}_i^k$ that is a weighted average of the observed slopes $(d_i^{j+1} - \beta_{i,0})/p_i^j$. That is,

$$\widehat{\alpha}_i^{k+1} \;:=\; \sum_{j=0}^{k} w^{j+1,k+1}\widetilde{\alpha}_i^{j+1}$$

where $w^{j+1,k+1}$ denotes the weight assigned to observed slope $\widetilde{\alpha}_i^{j+1} := (d_i^{j+1} - \beta_{i,0})/p_i^j$ when the observed data are $(p_i^0, d_i^1), \ldots, (p_i^k, d_i^{k+1})$. For example, the ordinary least squares estimator corresponds to $w^{j+1,k+1} = (p_i^j)^2/\sum_{\ell=0}^{k}(p_i^\ell)^2$. If $\{\varepsilon_i^k\}$ is i.i.d., then it can be shown that the ordinary least squares estimator has the demand consistency property. Here we consider the simpler estimator with $w^{j+1,k+1} = 1/(k+1)$ for all $j = 0, \ldots, k$, that is,

$$\widehat{\alpha}_i^{k+1} \;:=\; \frac{1}{k+1}\sum_{j=0}^{k}\widetilde{\alpha}_i^{j+1} \;=\; \frac{k}{k+1}\widehat{\alpha}_i^k + \frac{1}{k+1}\widetilde{\alpha}_i^{k+1}\,. \tag{24}$$

Our simulation experiments suggest that the limiting behavior of the estimators (24) is the same as that of the least squares estimator. We consider (24) rather than the least squares estimator for tractability. It follows from a strong law of large numbers for martingales (Chow 1967) that the estimator $\widehat{\alpha}_i^k$ in (24) has the demand consistency property, and that, if the sellers' models were correct, then $\widehat{\alpha}_i^k$ would be a consistent estimator for α_i as long as the seller chose the prices p_i^k to be asymptotically bounded away from zero.

It follows from (20), (24), and $d_i^{k+1} = \beta_{i,0} + \beta_{i,i}p_i^k + \beta_{i,-i}p_{-i}^k + \varepsilon_i^{k+1}$ that

$$\widehat{\alpha}_i^{k+1} \quad = \quad \widehat{\alpha}_i^k + \frac{1}{k+1}\left(\beta_{i,i} + \beta_{i,-i}\,\frac{\beta_{-i,0}}{\beta_{i,0}}\,\frac{\widehat{\alpha}_i^k}{\widehat{\alpha}_{-i}^k} - 2\frac{\widehat{\alpha}_i^k}{\beta_{i,0}}\varepsilon_i^{k+1} - \widehat{\alpha}_i^k\right) . \tag{25}$$

Note that $\widehat{\alpha}_i^{k+1}$ in (25) can take any real value. However, it follows from (20) that (i) positive slope estimates yield negative prices, and (ii) slope estimates equal to zero yield infinite prices. To avoid such values, suppose that each seller i chooses a number $b_i > 0$, and whenever $\widehat{\alpha}_i^{k+1}$ in (25) is greater than $-b_i$, the slope estimate is set equal to $-b_i$. We will say that these adjustments correspond to a "projection onto the $-b$-lower quadrant" of $\mathbb{R}^2$. Observe that such projections do not require any coordination between the sellers. Let $\widehat{\alpha}^k := (\widehat{\alpha}_{-1}^k, \widehat{\alpha}_1^k)$ and $\varepsilon^k := (\varepsilon_{-1}^k, \varepsilon_1^k)$. Let

$$H_i^{k+1} \quad := \quad \beta_{i,i} + \beta_{i,-i}\,\frac{\beta_{-i,0}}{\beta_{i,0}}\,\frac{\widehat{\alpha}_i^k}{\widehat{\alpha}_{-i}^k} - 2\frac{\widehat{\alpha}_i^k}{\beta_{i,0}}\varepsilon_i^{k+1} - \widehat{\alpha}_i^k \tag{26}$$

denote the direction of movement in $\widehat{\alpha}_i$ before projection after period $k+1$, and let $H^k := (H_{-1}^k, H_1^k)$. For each $x_i \in \mathbb{R}$, let $\Pi_i(x_i) := \min\{x_i, -b_i\}$ denote the projection used by seller i, and for $x = (x_{-1}, x_1)$, let $\Pi(x) := (\Pi_{-1}(x_{-1}), \Pi_1(x_1))$. Note that the component-wise projection $\Pi(x)$ coincides with the projection onto the $-b$-lower quadrant according to the usual Euclidean norm on $\mathbb{R}^2$. Thus, we consider slope estimates that satisfy

$$\widehat{\alpha}_i^{k+1} \quad := \quad \Pi_i\left(\widehat{\alpha}_i^k + \frac{1}{k+1}H_i^{k+1}\right). \tag{27}$$

Assume that each seller starts with an initial estimate $\widehat{\alpha}_i^0 \leq -b_i$. Let $\mathcal{F}^k$ denote the σ-field generated by $\widehat{\alpha}^0, \varepsilon^1, \ldots, \varepsilon^k$, and let

$$h_i^k \quad := \quad \mathbb{E}[H_i^{k+1}|\mathcal{F}^k] \quad = \quad \beta_{i,i} + \beta_{i,-i}\,\frac{\beta_{-i,0}}{\beta_{i,0}}\,\frac{\widehat{\alpha}_i^k}{\widehat{\alpha}_{-i}^k} - \widehat{\alpha}_i^k \tag{28}$$

denote the conditional expected direction of movement in the estimate of seller i (before projection) from point $\widehat{\alpha}_i^k$. For any $x = (x_{-1}, x_1) \in \mathbb{R}^2$, let

$$h_i(x) \quad := \quad \beta_{i,i} + \beta_{i,-i}\,\frac{\beta_{-i,0}}{\beta_{i,0}}\,\frac{x_i}{x_{-i}} - x_i \quad , \tag{29}$$

let $h(x) := (h_{-1}(x), h_1(x))$, and let $h^k := (h^k_{-1}, h^k_1)$. Note that $h^k = h(\widehat{\alpha}^k)$. Let $\bar{\beta}^* := (\beta^*_{-1}, \beta^*_1)$, where β^*_i is given by (21). Note that $\bar{\beta}^*$ is the unique solution of $h(\alpha) = 0$.

Next we restrict our attention to symmetric settings where $\beta_{i,0} = \beta_0$, $\beta_{i,i} = \beta_e$, and $\beta_{i,-i} = \beta_d$ for $i = \pm 1$. Consistent with (3), we assume $\beta_0 > 0$, $\beta_e < 0 \leq \beta_d$, and $|\beta_e| > \beta_d$. Note that by Proposition 2, in the symmetric case the MEE is Pareto superior to the NE when $\beta_d > 0$. (If $\beta_d = 0$ then the two notions of equilibria coincide.) To see this, observe that the pair $(\beta_{-1,-1}, \beta_{1,1}) = (\beta_e, \beta_e)$ lies on the diagonal line in $\mathcal{S}$, which is in $\Gamma^*(\theta)$ for any $\theta \in (0,1)$ because $\rho_i = 1$ for symmetric sellers and $T(\theta) > 1$. Under these symmetry assumptions, (26) becomes

$$H^{k+1}_i \quad := \quad \beta_e + \beta_d \frac{\widehat{\alpha}^k_i}{\widehat{\alpha}^k_{-i}} - 2\frac{\widehat{\alpha}^k_i}{\beta_0}\varepsilon^{k+1}_i - \widehat{\alpha}^k_i \tag{30}$$

For $x = (x_{-1}, x_1)$ the expression (29) becomes

$$h_i(x) \quad := \quad \beta_e + \beta_d \frac{x_i}{x_{-i}} - x_i \quad = \quad \beta_e + \beta_d - x_i + \beta_d\left(\frac{x_i}{x_{-i}} - 1\right). \tag{31}$$

Moreover, β^*_i defined in (21) simplifies to $\beta^*_i = \beta^* := \beta_e + \beta_d$ so that $\bar{\beta}^* = (\beta^*, \beta^*)$.

The main result of this section is Theorem 1 below which shows that w.p.1, the iterates $\widehat{\alpha}^k_i$ converge to β^* as $k \to \infty$. Although we do not have a proof of convergence for cases with asymmetric sellers, simulations suggest that there is indeed convergence of $\{\widehat{\alpha}^k\}$ to $\bar{\beta}^*$. Note that for the iterates $\widehat{\alpha}^k_i$ to converge to β^*, it should hold that $\bar{\beta}^*$ lies in the $-b$-lower quadrant, i.e., $\beta^* \leq -b_i$ for $i = \pm 1$. To establish the result we shall impose additional assumptions on b_i for $i = \pm 1$. The assumptions hold, for instance, when $\beta^* \leq -b_{-1} = -b_1$. Let $b_{\min} := \min\{b_{-1}, b_1\}$ and $b_{\max} := \max\{b_{-1}, b_1\}$.

Theorem 1. *Consider the sequence $\{\widehat{\alpha}^k\}$ given by (30) and (27). Suppose that $\beta^* \leq -b_{\max}$, and that*

$$\left(1 + \frac{b_{\min}}{\beta^*}\right)^2 \quad \leq \quad 1 + \frac{b_{\max}}{\beta^*}. \tag{32}$$

Assume that $\mathbb{E}[\varepsilon^{k+1}_i | \mathcal{F}^k] = 0$, and there is an M such that $\mathbb{E}[(\varepsilon^{k+1}_i)^2 | \mathcal{F}^k] \leq M$, w.p.1 for all k and for each i. Then, w.p.1, for each i,

$$\lim_{k\to\infty} \widehat{\alpha}^k_i \quad = \quad \beta^*.$$

Figure 3 depicts the convergence described in the theorem for a deterministic system ($\varepsilon^k_i \equiv 0$). Figure 4 shows a sample path with i.i.d. normal noise. Note that the limit point $\bar{\beta}^*$ lies at the intersection of two curves. One of the curves shows points x where $h_{-1}(x) = 0$, that is, points $\widehat{\alpha}$ at which the expected direction of movement of the estimate $\widehat{\alpha}_{-1}$ of seller -1 is 0. The other curve shows points x where $h_1(x) = 0$. The intersection of the curves is the point $\bar{\beta}^*$ at which $h(\bar{\beta}^*) = 0$.

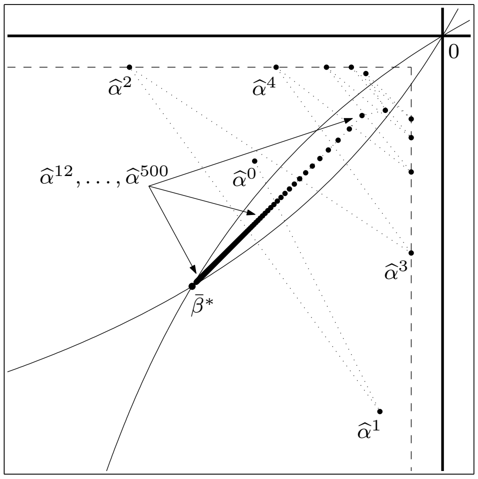

Figure 3: Trajectory of 500 iterates of $\{\widehat{\alpha}^k := (\widehat{\alpha}^k_{-1}, \widehat{\alpha}^k_1)\}$ of a deterministic process ($\varepsilon_i^k \equiv 0$) with $\beta_0 = 1$, $\beta_e = -1$, $\beta_d = 0.6$, $b_{-1} = b_1 = 0.05$, $\widehat{\alpha}^0 = (-0.3, -0.2)$. The slope estimates $\{\widehat{\alpha}^k\}$ converge to $\bar{\beta}^* = (-0.4, -0.4)$.

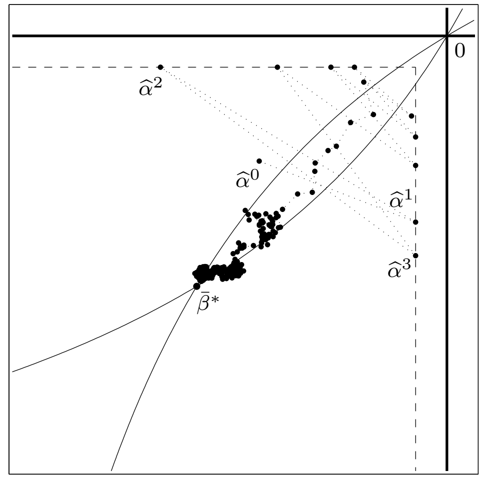

Figure 4: Trajectory of 500 iterates of $\{\widehat{\alpha}^k := (\widehat{\alpha}^k_{-1}, \widehat{\alpha}^k_1)\}$ of a random process (ε_i^k i.i.d. normal, mean 0, standard deviation 0.5) with $\beta_0 = 1$, $\beta_e = -1$, $\beta_d = 0.6$, $b_{-1} = b_1 = 0.05$, $\widehat{\alpha}^0 = (-0.3, -0.2)$. The slope estimates $\{\widehat{\alpha}^k\}$ converge to $\bar{\beta}^* = (-0.4, -0.4)$.

It follows from Theorem 1 and (20) that w.p.1,

$$\lim_{k\to\infty} p_i^k \quad = \quad -\frac{\beta_0}{2(\beta_e + \beta_d)} \qquad i \pm 1, \tag{33}$$

so that the limit prices coincide with the cooperative solution (8). [Note that (8) simplifies to the right side of (33) in the case of symmetric sellers; i.e., when $\beta_{i,0} = \beta_0$, $\beta_{i,i} = \beta_e$, and $\beta_{i,-i} = \beta_d$ for $i = \pm 1$.] If each seller uses an incorrect model that neglects the effect of its competitor, and thinks that it is estimating the sensitivity of its own demand to its own price, then in the limit the sellers settle on prices that are *not* the Nash equilibrium associated with the correct model, and that in fact maximize their combined revenue in the symmetric case. [In the Nash equilibrium in the symmetric case, each price is equal to $-\beta_0/(2\beta_e + \beta_d)$, which is less than the limit price in (33).] This is markedly different behavior than that which arises in a setting where each seller employs a correct model with known parameters and must adjust its prices as it tries to learn about its competitor's prices (e.g., Cournot adjustment or fictitious play). As mentioned in Section 3, the sellers' prices converge to the Nash equilibrium under both Cournot adjustment and fictitious play. It also differs from the behavior in Section 4.1, where prices also converge to the Nash equilibrium.

Intuition for the Theorem and Intermediate Results. The proof of Theorem 1 uses several lemmas, some of which follow in the main text and some of which appear in the appendix. Here we begin by providing some insight into one key piece of our argument that establishes that the expected direction of movement of the iterates is — in a sense to be described below — toward $\bar{\beta}^*$. This portion of the argument requires particular care and relies on understanding the dynamics of the iterates.

First we introduce a few definitions. Let $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^2$; i.e., $\|x\| = (x_{-1}^2 + x_1^2)^{1/2}$. Let $\mathcal{D} := \{y \in \mathbb{R}^2 : y_{-1} = y_1\}$ denote the diagonal line, and for $x \in \mathbb{R}^2$ let $\mathfrak{d}(x) := \min\{\|x - y\| : y \in \mathcal{D}\}$ denote the Euclidean distance from x to the diagonal line. Observe that

$$[\mathfrak{d}(x)]^2 = \left\|(x_{-1}, x_1) - \left(\frac{x_{-1}+x_1}{2}, \frac{x_{-1}+x_1}{2}\right)\right\|^2 = \frac{1}{2}(x_1 - x_{-1})^2 . \tag{34}$$

Define

$$Q := \begin{bmatrix} 1 & -q \\ -q & 1 \end{bmatrix} \tag{35}$$

for $q \in [0, 1)$, so that Q is positive definite. For any $x \in \mathbb{R}^2$, let $\|x\|_Q$ denote the Q-norm of x, i.e. $\|x\|_Q := \sqrt{x^T Q x}$. Note that

$$\|x\|_Q^2 = x_{-1}^2 + x_1^2 - 2q x_{-1} x_1 = (1-q)\|x\|^2 + q2[\mathfrak{d}(x)]^2 . \tag{36}$$

Thus, the squared Q-norm of $x \in \mathbb{R}^2$ is a convex combination of the squared Euclidean norm of x and twice the squared Euclidean distance from x to the diagonal line $\mathcal{D}$.

The Q-norm is simply a tool we use to prove Theorem 1; it has no effect on the prices or the sellers' behavior. To understand the use of the Q-norm, refer to Figure 5. In the figure, the dotted concentric ellipses around $\bar{\beta}^*$ correspond to points that are equidistant from $\bar{\beta}^*$, as measured by the Q-norm with $q = 0.5$. That is, any two points x and y on a single ellipse satisfy $\|\bar{\beta}^* - x\|_Q = \|\bar{\beta}^* - y\|_Q$. If $q = 0$, then the Q-norm reduces to the usual Euclidean norm, and the ellipses that represent points equidistant to $\bar{\beta}^*$ are circles centered at $\bar{\beta}^*$. As the value of q increases, the ellipses are "stretched" in the direction of the 45 degree line. For any two points x and y such that (a) they are equidistant from $\bar{\beta}^*$ as measured by the usual Euclidean norm

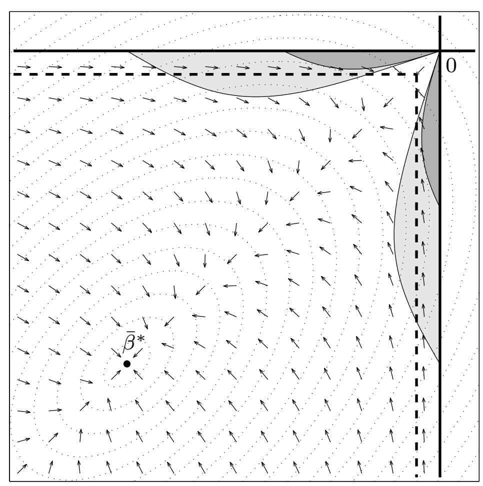

Figure 5: The Q-norm with $q = 0.5$ and expected directions of movement.

and (b) x is closer than y to the diagonal, the Q-norm will measure x to be closer than y to $\bar{\beta}^*$; i.e., $\|\bar{\beta}^* - x\|_Q \leq \|\bar{\beta}^* - y\|_Q$. This reflects the idea that iterates close to the diagonal are close to $\bar{\beta}^*$, consistent with Figures 3 and 4.

The arrows in Figure 5 represent the expected directions $h(\cdot)$ given in (31); all arrows have been scaled to the same size so that they show the direction, but not the magnitude of $h(\cdot)$. The shaded regions (both dark and light) correspond to those points in $\mathbb{R}^2_-$ such that the direction of movement is away, in the sense of Euclidean distance, from $\bar{\beta}^*$. Note that there are points in the $-b$-quadrant (the region "southwest" of the heavy dashed lines) for which the direction of movement, as measured by Euclidean distance, is away from $\bar{\beta}^*$. Hence, for this particular example (and many others), it is not true that the expected direction of movement of the iterates is always toward the limit point in the sense of the Euclidean inner product. The dark shaded regions in Figure 5 correspond to those points in $\mathbb{R}^2_-$ such that the direction of movement is away, in the sense of the Q-norm, from $\bar{\beta}^*$. For such points, the arrows in the figure would point from smaller ellipses to larger ellipses. (To avoid clutter only a few such arrows are shown.) The light shaded region contains those points in $\mathbb{R}^2_-$ such that the direction of movement is (i) away from $\bar{\beta}^*$ when distance is measured by the Euclidean norm, but (ii) towards $\bar{\beta}^*$ when distance is measured by the Q-norm with $q = 0.5$. A key point is that as long as $\beta^* \leq -b_{\max}$ and some conditions hold, it is always possible to ensure that the dark shaded regions lie entirely outside the $-b$-quadrant (in the figure, the dark region is "north" and "east" of the heavy dashed lines) by taking q in (35) close enough to 1. (In this particular example, taking $q = 0.5$ suffices.) This is made precise in Lemma 2 below.

Lemma 1 shows that the projection Π moves points closer in Q-norm to $\bar{\beta}^*$. The result would be simple if Π was the projection according to the Q-norm rather than component-wise projection.

Lemma 1. *Suppose that $\beta^* \leq -b_{\max}$ and either $\beta^* + b_{\min} = 0$ or*

$$q \quad \leq \quad 1 + \frac{b_{\max} - b_{\min}}{\beta^* + b_{\min}}. \tag{37}$$

Then

$$\|\Pi(x) - \bar{\beta}^*\|_Q \quad \leq \quad \|x - \bar{\beta}^*\|_Q \qquad \textit{for all } \; x \in \mathbb{R}^2. \tag{38}$$

The next lemma shows that the expected direction of movement h^k (without projection) makes an acute angle, in the sense of the inner product $\langle u, v\rangle := u^T Q v$, with the direction from $\widehat{\alpha}^k$ to $\bar{\beta}^*$. Put differently, the expected direction of movement, as measured by the Q-norm, is toward $\bar{\beta}^*$.

Lemma 2. *Suppose that* $\beta^* \le -b_{\max}$*, and that* $q \in [1 + b_{\min}/\beta^*, 1)$*. Consider any point* $x = (x_{-1}, x_1)$ *in the* $-b$*-lower quadrant. Let* $y := \bar{\beta}^* - x$ *denote the direction from* x *to* $\bar{\beta}^*$*. Then*

$$h(x)^T Q y \quad \ge \quad \|y\|_Q^2 \, .$$

Proof of Theorem 1. We begin by showing that we can choose q so that Lemmas 1 and 2 can be applied. Lemma 1 required either $\beta^* + b_{\min} = 0$ or $q \le 1 + (b_{\max} - b_{\min})/(\beta^* + b_{\min})$, and Lemma 2 required $q \ge 1 + b_{\min}/\beta^*$. Thus we require either that $\beta^* + b_{\min} = 0$ and $q \in [0, 1)$, or that

$$q \quad \in \quad \left[1 + \frac{b_{\min}}{\beta^*}, 1 + \frac{b_{\max} - b_{\min}}{\beta^* + b_{\min}}\right].$$

Next we show that the interval above is a nonempty subset of $[0, 1)$ when $\beta^* + b_{\min} < 0$. Note that $1 + b_{\min}/\beta^* \in [0, 1)$ and $1 + (b_{\max} - b_{\min})/(\beta^* + b_{\min}) \in [0, 1)$. It is easy to check that (32) is equivalent to $(b_{\min})^2 \le -\beta^* \, (2b_{\min} - b_{\max})$. This and $\beta^* + b_{\min} < 0$ imply that

$$1 + \frac{b_{\min}}{\beta^*} \quad = \quad 1 + \frac{\beta^* b_{\min} + (b_{\min})^2}{\beta^*(\beta^* + b_{\min})} \quad \le \quad 1 + \frac{-\beta^* \, (b_{\min} - b_{\max})}{\beta^*(\beta^* + b_{\min})} \quad = \quad 1 + \frac{b_{\max} - b_{\min}}{\beta^* + b_{\min}}.$$

Thus (32) and $\beta^* + b_{\min} < 0$ imply that the interval $[1 + b_{\min}/\beta^*, 1 + (b_{\max} - b_{\min})/(\beta^* + b_{\min})]$ of considered values for q is a nonempty subset of $[0, 1)$. Hence, we can choose q so that Lemmas 1 and 2 can be applied.

Next, let $Z^k := \left\|\bar{\beta}^* - \widehat{\alpha}^k\right\|_Q^2$. Then it follows from (27) and Lemma 1 that

$$\begin{aligned}
Z^{k+1} \quad &= \quad \left\|\bar{\beta}^* - \Pi\left(\widehat{\alpha}^k + \frac{1}{k+1} H^{k+1}\right)\right\|_Q^2 \\
&\le \quad \left\|\bar{\beta}^* - \left(\widehat{\alpha}^k + \frac{1}{k+1} H^{k+1}\right)\right\|_Q^2 \\
&= \quad Z^k - \frac{2}{k+1} (H^{k+1})^T Q \left(\bar{\beta}^* - \widehat{\alpha}^k\right) + \left(\frac{1}{k+1}\right)^2 (H^{k+1})^T Q H^{k+1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}\left[Z^{k+1} \,|\, \mathcal{F}^k\right] \quad \le \quad & Z^k - \frac{2}{k+1} \mathbb{E}\left[H^{k+1} \,\middle|\, \mathcal{F}^k\right]^T Q \left(\bar{\beta}^* - \widehat{\alpha}^k\right) \\
& + \left(\frac{1}{k+1}\right)^2 \mathbb{E}\left[(H^{k+1})^T Q H^{k+1} \,\middle|\, \mathcal{F}^k\right] .
\end{aligned} \tag{39}$$

It follows from Lemma 2 that

$$\mathbb{E}\left[H^{k+1} \,\middle|\, \mathcal{F}^k\right]^T Q \left(\bar{\beta}^* - \widehat{\alpha}^k\right) \quad = \quad (h^k)^T Q \left(\bar{\beta}^* - \widehat{\alpha}^k\right) \quad \ge \quad \left\|\bar{\beta}^* - \widehat{\alpha}^k\right\|_Q^2 . \tag{40}$$

In addition, Lemma A–2 in the Appendix establishes the existence of constants A and B such that $\mathbb{E}\left[(H^{k+1})^T Q H^{k+1} \mid \mathcal{F}^k\right] \le A + B \left\|\bar{\beta}^* - \widehat{\alpha}^k\right\|_Q^2$. Using this fact and (40), it follows from (39) that

$$\begin{aligned}
\mathbb{E}\left[Z^{k+1} \,|\, \mathcal{F}^k\right] &\le Z^k - \frac{2}{k+1}\left\|\bar{\beta}^* - \widehat{\alpha}^k\right\|_Q^2 + \left(\frac{1}{k+1}\right)^2 \left(A + B\left\|\bar{\beta}^* - \widehat{\alpha}^k\right\|_Q^2\right) \\
&= \left[1 + \left(\frac{1}{k+1}\right)^2 B\right] Z^k + \left(\frac{1}{k+1}\right)^2 A - \frac{2}{k+1} Z^k.
\end{aligned}$$

Finally, by applying Lemma A–3 in the Appendix with

$$B^k = \left(\frac{1}{k+1}\right)^2 B, \qquad C^k = \left(\frac{1}{k+1}\right)^2 A, \qquad D^k = \frac{2}{k+1} Z^k,$$

it follows that there is a finite random variable Z such that w.p.1, $Z^k \to Z$ and $\sum_k D^k < \infty$. Next we show that $Z = 0$ w.p.1. Consider any ω such that $Z(\omega) \neq 0$, i.e., $Z(\omega) > 0$. Then, for all k sufficiently large, $Z^k(\omega) > Z(\omega)/2$, which implies that $\sum_k D^k(\omega) = \infty$. Hence, $Z = 0$ w.p.1. □

4.3 The Case with Unknown Intercept and Unknown Slope

In this section, we consider the situation in which each seller i assumes that demand has the form (9), and estimates *both* parameters $\alpha_{i,0}$ and α_i.

Demand Consistency. It can be shown that for any pair of prices $(p_{-1}^\infty, p_1^\infty)$ such that $p_i^\infty \neq 0$ and $d_i(p_i^\infty, p_{-i}^\infty) \neq 0$ for $i = \pm 1$, there exist parameter estimates $(\widehat{\alpha}_{i,0}^\infty, \widehat{\alpha}_i^\infty) = (2d_i(p_i^\infty, p_{-i}^\infty), -d_i(p_i^\infty, p_{-i}^\infty)/p_i^\infty)$, $i = \pm 1$, such that the estimated demand and the expected demand at prices $(p_{-1}^\infty, p_1^\infty)$ are equal, and the prices p_i^∞ are consistent with the parameter estimates $(\widehat{\alpha}_{i,0}^\infty, \widehat{\alpha}_i^\infty)$, $i = \pm 1$, that is, $\widehat{\delta}_i^\infty(p_i^\infty) = \widehat{\alpha}_{i,0}^\infty + \widehat{\alpha}_i^\infty p_i^\infty = d_i(p_i^\infty, p_{-i}^\infty) = \beta_{i,0} + \beta_{i,i} p_i^\infty + \beta_{i,-i} p_{-i}^\infty$ and $p_i^\infty = -\widehat{\alpha}_{i,0}^\infty/(2\widehat{\alpha}_i^\infty)$ for $i = \pm 1$. Thus, in this case, the demand consistency property allows infinitely many potential limit points.

Convergence. In this section we consider the ordinary least squares estimates. To simplify the analysis, throughout this section we assume that $\beta_{i,-i} > 0$ for $i = \pm 1$, and also that demand is deterministic, i.e., $\varepsilon_i^k \equiv 0$.

After period k, the estimated demand model of seller i is

$$\widehat{\delta}_i^k(p_i) = \widehat{\alpha}_{i,0}^k + \widehat{\alpha}_i^k p_i \tag{41}$$

where $\widehat{\alpha}_{i,0}^k$ and $\widehat{\alpha}_i^k$ are the least squares estimates with the data pairs $(p_i^0, d_i^1), \ldots, (p_i^{k-1}, d_i^k)$. Seller i then chooses price

$$p_i^k = -\frac{\widehat{\alpha}_{i,0}^k}{2\widehat{\alpha}_i^k} \tag{42}$$

as long as $\widehat{\alpha}_i^k < 0$; see (11). If $\widehat{\alpha}_i^k \geq 0$ then the problem $\max_{p_i} p_i \widehat{\delta}_i^k(p_i)$ is unbounded and then the iterative procedure ends. Here, unlike in the previous section, we do not consider projected slope estimates. Next, each seller i observes its demand

$$d_i^{k+1} \quad = \quad d_i(p_i^k, p_{-i}^k) \quad = \quad \beta_{i,0} + \beta_{i,i} p_i^k + \beta_{i,-i} p_{-i}^k \,. \tag{43}$$

Thereafter, each seller updates its least squares estimate of (41), and the process repeats.

Let us study the dynamics of p_i^k and d_i^k. Suppose that each seller i has observed price-quantity pairs $(p_i^0, d_i^1), \ldots, (p_i^k, d_i^{k+1})$ by the end of period $k+1$. First we write p_i^{k+1} as a function of these past observations. To do so, it is useful to define the following quantities:

$$\begin{aligned}
\overline{p}_i^{k+1} &:= \frac{1}{k+1} \sum_{\ell=0}^{k} p_i^\ell \quad = \quad \frac{k}{k+1} \overline{p}_i^k + \frac{1}{k+1} p_i^k \\
\overline{d}_i^{k+1} &:= \frac{1}{k+1} \sum_{\ell=1}^{k+1} d_i^\ell \quad = \quad \frac{k}{k+1} \overline{d}_i^k + \frac{1}{k+1} d_i^{k+1} \\
\overline{p^2}_i^{k+1} &:= \frac{1}{k+1} \sum_{\ell=0}^{k} (p_i^\ell)^2 \quad = \quad \frac{k}{k+1} \overline{p^2}_i^k + \frac{1}{k+1} (p_i^k)^2 \\
\overline{pd}_i^{k+1} &:= \frac{1}{k+1} \sum_{\ell=1}^{k+1} p_i^{\ell-1} d_i^\ell \quad = \quad \frac{k}{k+1} \overline{pd}_i^k + \frac{1}{k+1} p_i^k d_i^{k+1} \\
\overline{pp}^{k+1} &:= \frac{1}{k+1} \sum_{\ell=0}^{k} p_i^\ell p_{-i}^\ell,
\end{aligned}$$

where p_i^k is given by (42) and d_i^{k+1} is given by (43). It follows from (43) that

$$\begin{aligned}
\overline{d}_i^{k+1} &= \beta_{i,0} + \beta_{i,i} \overline{p}_i^{k+1} + \beta_{i,-i} \overline{p}_{-i}^{k+1} \\
\overline{pd}_i^{k+1} &= \beta_{i,0} \overline{p}_i^{k+1} + \beta_{i,i} \overline{p^2}_i^{k+1} + \beta_{i,-i} \overline{pp}^{k+1}.
\end{aligned}$$

Moreover, the familiar normal equations for linear regression yield

$$\widehat{\alpha}_i^k \quad = \quad \frac{\overline{pd}_i^k - \overline{p}_i^k \overline{d}_i^k}{\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2} \quad = \quad \beta_{i,i} + \beta_{i,-i} \frac{\overline{pp}^k - \overline{p}_i^k \overline{p}_{-i}^k}{\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2} \tag{44}$$

$$\widehat{\alpha}_{i,0}^k \quad = \quad \overline{d}_i^k - \widehat{\alpha}_i^k \overline{p}_i^k \quad = \quad \beta_{i,0} + \beta_{i,-i} \frac{\overline{p}_{-i}^k \overline{p^2}_i^k - \overline{p}_i^k \overline{pp}^k}{\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2} \tag{45}$$

and then we can use (42) to express p_i^k as a function of $\overline{p}_i^k$, $\overline{p}_{-i}^k$, $\overline{p^2}_i^k$, and $\overline{pp}^k$.

Next, let

$$r_i^k \quad := \quad \frac{\overline{pp}^k - \overline{p}_i^k \overline{p}_{-i}^k}{\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2} \tag{46}$$

denote the ratio in (44). Then, we can write the parameter estimates in (44) and (45) as

$$\widehat{\alpha}_i^k = \beta_{i,i} + \beta_{i,-i} r_i^k \tag{47}$$

$$\widehat{\alpha}_{i,0}^k = \beta_{i,0} + \beta_{i,-i}\left(\overline{p}_{-i}^k - \overline{p}_i^k r_i^k\right). \tag{48}$$

Similarly, we can write the prices p_i^k as

$$p_i^k = -\frac{\widehat{\alpha}_{i,0}^k}{2\widehat{\alpha}_i^k} = -\frac{\beta_{i,0} + \beta_{i,-i}\left(\overline{p}_{-i}^k - \overline{p}_i^k r_i^k\right)}{2\left(\beta_{i,i} + \beta_{i,-i} r_i^k\right)}. \tag{49}$$

A useful interpretation of r_i^k is as follows: for a fixed k, let (X_{-1}, X_1) denote a bivariate random vector that takes the values of the k empirically observed price pairs $\{(p_{-1}^0, p_1^0), \ldots, (p_{-1}^{k-1}, p_1^{k-1})\}$ each with probability $1/k$. Then

$$r_i^k = \frac{\mathrm{Cov}^k(X_i, X_{-i})}{\mathrm{Var}^k(X_i)}, \tag{50}$$

where the expectations in Cov^k and Var^k are taken with respect to the empirical measure introduced immediately above (50). Note that the expression in the denominator of (46) is $\mathrm{Var}^k(X_i)$, which is equal to zero if and only if $p_i^0 = \cdots = p_i^{k-1}$. Therefore we assume that the initial prices satisfy $p_i^0 \neq p_i^1$, which ensures that r_i^k is well defined for all k.

Also note that

$$r_{-1}^k r_1^k = \frac{\left[\mathrm{Cov}^k(X_{-1}, X_1)\right]^2}{\mathrm{Var}^k(X_{-1})\mathrm{Var}^k(X_1)} = \left[\mathrm{Corr}^k(X_{-1}, X_1)\right]^2$$

and hence

$$0 \leq r_{-1}^k r_1^k \leq 1 \quad \text{for all } k. \tag{51}$$

Moreover, it follows that $r_{-1}^k r_1^k = 1$ if and only if all k pairs $(p_{-1}^0, p_1^0), \ldots, (p_{-1}^{k-1}, p_1^{k-1})$ lie on a straight line in $\mathbb{R}^2$. This, of course, is the case at $k = 2$. Also, $r_{-1}^k r_1^k = 0$ if and only if $r_{-1}^k = 0$ and $r_1^k = 0$ — this follows from the fact that $r_i^k = 0 \Leftrightarrow \mathrm{Cov}^k(X_{-1}, X_1) = 0$.

Equation (49) relates the prices p_i^k to the empirical mean, variance, and covariance of the previous prices. It follows from (49) that

$$r_i^k = \frac{\beta_{i,0} + \beta_{i,-i}\overline{p}_{-i}^k + 2\beta_{i,i} p_i^k}{\beta_{i,-i}\left(\overline{p}_i^k - 2p_i^k\right)}. \tag{52}$$

To study the asymptotic behavior of p_i^k, we first characterize potential limits. Suppose that the prices (p_{-1}^k, p_1^k) converge to some limit $(p_{-1}^*, p_1^*) > 0$ as $k \to \infty$. Then it follows from (52) that

$$r_i^k \to r_i^* = \frac{\beta_{i,0} + \beta_{i,-i} p_{-i}^* + 2\beta_{i,i} p_i^*}{-\beta_{i,-i} p_i^*} \tag{53}$$

for $i = \pm 1$. It follows from (51) that

$$0 \quad \leq \quad r^*_{-1} r^*_1 \quad \leq \quad 1 \tag{54}$$

Moreover, the condition $\widehat{\alpha}^k_i < 0$ for all k and (47) imply that

$$r^*_i \quad \leq \quad \frac{-\beta_{i,i}}{\beta_{i,-i}}, \qquad i = \pm 1. \tag{55}$$

Note that (53) can be written as a linear system in (p^*_{-1}, p^*_1):

$$\beta_{i,-i} p^*_{-i} + \left(\beta_{i,-i} r^*_i + 2\beta_{i,i}\right) p^*_i \quad = \quad -\beta_{i,0}\,, \qquad i = \pm 1. \tag{56}$$

As shown in the appendix, if (55) holds, then the linear system is nonsingular, and yields the following solution for (p^*_{-1}, p^*_1):

$$p^*_i \quad = \quad p_i(r^*_{-1}, r^*_1) \tag{57}$$

where

$$p_i(r_{-1}, r_1) \quad := \quad \frac{-2\,\beta_{i,0}\,\beta_{-i,-i} + \beta_{-i,0}\,\beta_{i,-i} - \beta_{i,0}\,\beta_{-i,i}\,r_{-i}}{4\,\beta_{-i,-i}\,\beta_{i,i} + 2\,\beta_{-i,-i}\,\beta_{i,-i}\,r_i + 2\,\beta_{i,i}\,\beta_{-i,i}\,r_{-i} - \beta_{-i,i}\,\beta_{i,-i}\,(1 - r_{-i}\,r_i)}$$

Let

$$\begin{aligned} \mathcal{R} \quad &:= \quad \left\{ (r_{-1}, r_1) \;:\; 0 \leq r_{-1} r_1 \leq 1,\; r_i \leq \frac{-\beta_{i,i}}{\beta_{i,-i}},\; i = \pm 1 \right\} \\ \mathcal{P} \quad &:= \quad \{ (p_{-1}(r_{-1}, r_1), p_1(r_{-1}, r_1)) \;:\; (r_{-1}, r_1) \in \mathcal{R} \} \end{aligned}$$

that is, $\mathcal{R}$ is the set of all pairs (r_{-1}, r_1) that satisfy (54) and (55), and $\mathcal{P}$ is the set of potential limit prices. It is shown in the appendix that $p_i(r_{-1}, r_1) > 0$ for all $(r_{-1}, r_1) \in \mathcal{R}$, and thus $(p_{-1}, p_1) > 0$ for all $(p_{-1}, p_1) \in \mathcal{P}$. Also, the steps for deriving (57) from (56) can be reversed, and thus it follows that for any $(p_{-1}, p_1) \in \mathcal{P}$ there exists a unique pair $(r_{-1}, r_1) \in \mathcal{R}$, that is, $p(r_{-1}, r_1) := (p_{-1}(r_{-1}, r_1), p_1(r_{-1}, r_1))$ is a bijection from $\mathcal{R}$ to $\mathcal{P}$. Thus, if such (p^*_{-1}, p^*_1) is the limit of (p^k_{-1}, p^k_1), then the unique pair $(r^*_{-1}, r^*_1) = p^{-1}(p^*_{-1}, p^*_1)$ that satisfies (56) must be the limit of (r^k_{-1}, r^k_1), and thus must satisfy (54) and (55). Proposition 3 below summarizes the characterization of potential limit points derived above.

Proposition 3. *Suppose that the prices (p^k_{-1}, p^k_1) converge to some limit (p^*_{-1}, p^*_1) as $k \to \infty$ and $p^*_i > 0$ for $i = \pm 1$. Then $(p^*_{-1}, p^*_1) \in \mathcal{P}$. Associated with (p^*_{-1}, p^*_1) there is a unique pair (r^*_{-1}, r^*_1) that satisfies (54), (55) and (56), and that is the limit of (r^k_{-1}, r^k_1) as $k \to \infty$.*

Important questions are: (1) do the prices converge, and if so, (2) is the limit (p^*_{-1}, p^*_1) uniquely determined by the β-parameters, or are multiple limits $(p^*_{-1}, p^*_1) \in \mathcal{P}$ possible, depending on the initial prices? Below we show that almost all points in $\mathcal{P}$ are possible limit points. Specifically, we show that for almost all points $(p^*_{-1}, p^*_1) \in \mathcal{P}$, there are initial conditions such that the prices become invariant and equal to (p^*_{-1}, p^*_1) after a finite number of periods. To demonstrate this, we introduce some terminology. We say the process is *stationary* at period k_0 if for both $i = \pm 1$ it holds that $r_i^k = r_i^{k_0}$ and $p_i^k = p_i^{k_0}$ for all $k \geq k_0$. For $k_0 \geq 2$ we shall use the following conditions:

$$\overline{p}_i^{k_0} = p_i^{k_0} \tag{58}$$

and

$$\frac{\overline{p}_{-i}^{k_0} - p_{-i}^{k_0}}{\overline{p}_i^{k_0} - p_i^{k_0}} = r_i^{k_0} \quad \text{and} \quad \overline{p}_i^{k_0} \neq p_i^{k_0}\,. \tag{59}$$

The following lemma gives necessary and sufficient conditions for the process to be stationary.

Lemma 3. *The process is stationary at period k_0 if and only if either (58) holds for both $i = \pm 1$ or (59) holds for both $i = \pm 1$. Moreover, if the process is stationary with $r_{-1}^{k_0} r_1^{k_0} < 1$, then (58) must necessarily hold for both $i = \pm 1$.*

Using Lemma 3, we show in Proposition 4 below that the process can become stationary at most of the potential limit prices $(p^*_{-1}, p^*_1) \in \mathcal{P}$. The only exceptions are the following:

1. The points $(p^*_{-1}, p^*_1) \in \mathcal{P}$ corresponding to (r^*_{-1}, r^*_1) such that $r^*_{-1} r^*_1 = 0$ but either $r^*_{-1} \neq 0$ or $r^*_1 \neq 0$. It follows from the comment after (51) that, if the system is stationary at period k_0 with $r_{-1}^{k_0} r_1^{k_0} = 0$, then it must hold that $r^*_{-1} = r_{-1}^{k_0} = r^*_1 = r_1^{k_0} = 0$.

2. The points $(p^*_{-1}, p^*_1) \in \mathcal{P}$ corresponding to (r^*_{-1}, r^*_1) such that $r^*_i = -\beta_{i,i}/\beta_{i,-i}$ for $i = -1$ or $i = 1$, since it follows from (47) that in those cases $\widehat{\alpha}_i^{k_0} = \beta_{i,i} + \beta_{i,-i} r_i^{k_0} = \beta_{i,i} + \beta_{i,-i} r^*_i = 0$ and hence the iterative procedure stops [cf. remark after (42)].

The exceptions above could be limit values but not stationary ones. Thus, condition (54) is replaced with

$$\text{either} \quad r^*_{-1} = r^*_1 = 0 \quad \text{or} \quad 0 < r^*_{-1} r^*_1 \leq 1 \tag{60}$$

and condition (55) is replaced with

$$r^*_i < \frac{-\beta_{i,i}}{\beta_{i,-i}}, \qquad i = \pm 1. \tag{61}$$

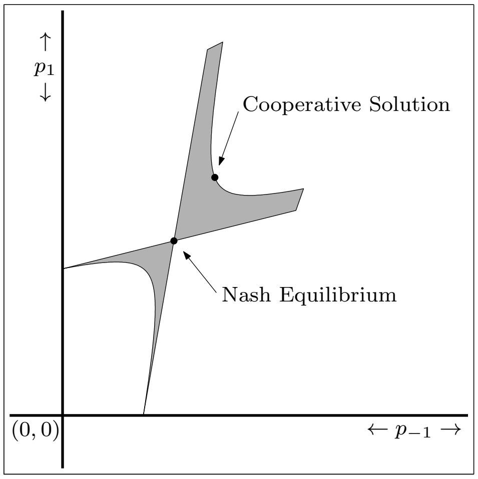

Figure 6: The shaded region is the set $\mathcal{P}'$ of stationary price pairs, $(\beta_{-1,0} = 1.1, \beta_{-1,-1} = -2.0, \beta_{-1,1} = 0.7, \beta_{1,0} = 1.0, \beta_{1,1} = -1.0, \beta_{1,-1} = 0.5)$.

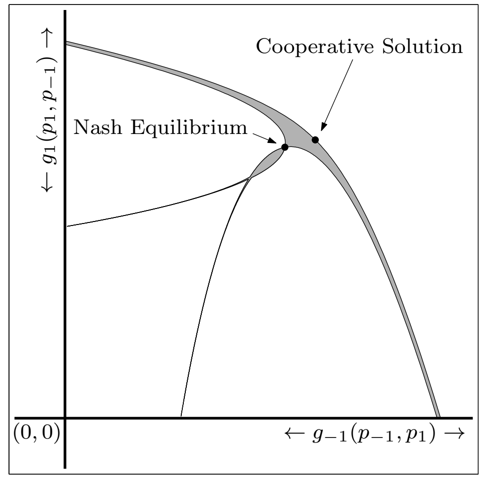

Figure 7: The shaded region is the set of stationary revenue pairs $\{(g_{-1}(p_{-1}, p_1), g_1(p_1, p_{-1})) : (p_{-1}, p_1) \in \mathcal{P}'\}$, $(\beta_{-1,0} = 1.1, \beta_{-1,-1} = -2.0, \beta_{-1,1} = 0.7, \beta_{1,0} = 1.0, \beta_{1,1} = -1.0, \beta_{1,-1} = 0.5)$.

Let

$$\begin{aligned} \mathcal{R}' &:= \big\{(r_{-1}, r_1) \in \mathbb{R}^2 \ : \ (60) \text{ and } (61) \text{ hold}\big\} \\ \mathcal{P}' &:= \big\{(p_{-1}(r_{-1}, r_1), p_1(r_{-1}, r_1)) \ : \ (r_{-1}, r_1) \in \mathcal{R}'\big\} \end{aligned}$$

Proposition 4. *For any $(p^*_{-1}, p^*_1) \in \mathcal{P}'$ there exist initial points (p^k_{-1}, p^k_1), $k = 0, 1, 2$, such that the process becomes stationary at $k_0 = 3$ with $p^{k_0}_i = p^*_i$, $i = \pm 1$.*

Kirman (1983, Proposition 4) proved a similar result for symmetric sellers, but did not cover the case analogous to our condition $r^*_{-1} = r^*_1 = 0$, which corresponds to the Nash equilibrium. [One can see that $r^*_{-1} = r^*_1 = 0$ yields the Nash equilibrium by plugging $r^*_{-1} = r^*_1 = 0$ into (57) and comparing with (6).] Brousseau and Kirman (1992) re-state Proposition 4 of Kirman (1983) and include without proof the Nash equilibrium as a stationary price pair.

For a particular example with asymmetric sellers, Figure 6 shows the set $\mathcal{P}'$ of stationary price pairs identified in Proposition 4. A similar figure for symmetric sellers appears in Kirman (1983). It is interesting to note that both the Nash equilibrium $(p^{\mathrm{N}}_{-1}, p^{\mathrm{N}}_1)$ and the cooperative solution $(p^{\mathrm{C}}_{-1}, p^{\mathrm{C}}_1)$ are stationary price pairs. Figure 7 shows the set of revenue pairs corresponding to the price pairs in Figure 6; i.e., it shows $\{(g_{-1}(p_{-1}, p_1), g_1(p_1, p_{-1})) : (p_{-1}, p_1) \in \mathcal{P}'\}$. It is interesting to note that there is a portion of that set in which the revenues are Pareto superior to those associated with the Nash equilibrium, as well as a portion in which the revenues are Pareto inferior to those associated with the Nash equilibrium.

Market Response Hypothesis. As before, we determine necessary conditions for the Market Response Hypothesis to hold. Suppose that the sellers' parameter estimates converge. Then it follows from (47) and (48) that

$$\begin{aligned} \widehat{\alpha}_i^{\infty} &= \beta_{i,i} + \beta_{i,-i} r_i^{\infty} \\ \widehat{\alpha}_{i,0}^{\infty} &= \beta_{i,0} + \beta_{i,-i} \left(p_{-i}^{\infty} - p_i^{\infty} r_i^{\infty} \right) . \end{aligned}$$

The monopoly demand model of seller i captures competition if $\widehat{\delta}_i^{\infty}(p_i) = d_i(p_i, p_{-i}(p_i))$ for all p_i, that is, if

$$\widehat{\alpha}_{i,0}^{\infty} + \widehat{\alpha}_i^{\infty} p_i \;=\; \beta_{i,0} + \beta_{i,-i} \left(p_{-i}^{\infty} - p_i^{\infty} r_i^{\infty} \right) + \left(\beta_{i,i} + \beta_{i,-i} r_i^{\infty} \right) p_i \;=\; \beta_{i,0} + \beta_{i,i} p_i + \beta_{i,-i} p_{-i}(p_i) \quad \text{for all } p_i .$$

Thus, unless $\beta_{i,-i} = 0$, it is necessary that

$$p_{-i}(p_i) \;=\; p_{-i}^{\infty} - p_i^{\infty} r_i^{\infty} + r_i^{\infty} p_i \quad \text{for all } p_i .$$

Next, consider the short-run reaction function $p_{-i}(p_i) = p_{-i}^{\infty}$. It follows that the monopoly demand model of seller i captures competition for the short-run reaction function if $r_i^{\infty}\left(p_i - p_i^{\infty}\right) = 0$ for all p_i, that is, $r_i^{\infty} = 0$. Next, recall that $r_i^{\infty} = 0$ for $i = \pm 1$ if and only if the limit prices $(p_{-1}^{\infty}, p_1^{\infty})$ are the Nash equilibrium prices $(p_{-1}^{\mathrm{N}}, p_1^{\mathrm{N}})$. Thus, the Market Response Hypothesis holds for the short-run reaction function if and only if the sellers' parameter estimates converge and their corresponding prices converge to the Nash equilibrium prices. Since the pair of Nash equilibrium prices is only one point in the set $\mathcal{P}$ of possible limit prices, it would seem to be a rare possibility for the Market Response Hypothesis to hold for the short-run reaction function.

Next, consider the long-run reaction function. As before, suppose that the sellers' parameter estimates converge, and that after reaching the limit, seller i repeatedly sets price p_i. Then, in the long run, the estimates of seller $-i$ will move away from $(\widehat{\alpha}_{-i,0}^{\infty}, \widehat{\alpha}_{-i}^{\infty})$. It is difficult to determine whether the estimates of seller $-i$ "re-converge" because the behavior of these estimates may depend on the subset of the data collected by seller $-i$ up to that point that is used in the estimation. In the remainder of this discussion we consider the case in which the estimates of seller $-i$ converge to values that give the correct slope and intercept of the expected demand of seller $-i$ when seller i fixes its price at p_i, that is, the estimates of seller $-i$ converge to

$$\begin{aligned} \bar{\alpha}_{-i}^{\infty} &= \beta_{-i,-i} \\ \bar{\alpha}_{-i,0}^{\infty} &= \beta_{-i,0} + \beta_{-i,i} p_i . \end{aligned}$$

(It is easy to see that the estimates of seller $-i$ converge to these values if seller $-i$ discards the old data obtained before seller i fixes its price at p_i.)

With the above estimates, the corresponding price of seller $-i$ is

$$p_{-i}(p_i) \quad = \quad -\frac{\bar{\alpha}^{\infty}_{-i,0}}{2\bar{\alpha}^{\infty}_{-i}} \quad = \quad -\frac{\beta_{-i,0} + \beta_{-i,i} p_i}{2\beta_{-i,-i}} \,. \tag{62}$$

The resulting demand of seller i is

$$d_i(p_i, p_{-i}(p_i)) \quad = \quad \beta_{i,0} + \beta_{i,i} p_i + \beta_{i,-i} p_{-i}(p_i) \quad = \quad \left(\beta_{i,0} - \frac{\beta_{-i,0}\beta_{i,-i}}{2\beta_{-i,-i}}\right) + \left(\beta_{i,i} - \frac{\beta_{-i,i}\beta_{i,-i}}{2\beta_{-i,-i}}\right) p_i$$

The monopoly demand model of seller i captures competition for the reaction function (62) if and only if

$$\widehat{\alpha}^{\infty}_{i,0} + \widehat{\alpha}^{\infty}_{i} p_i \quad = \quad \left(\beta_{i,0} - \frac{\beta_{-i,0}\beta_{i,-i}}{2\beta_{-i,-i}}\right) + \left(\beta_{i,i} - \frac{\beta_{-i,i}\beta_{i,-i}}{2\beta_{-i,-i}}\right) p_i \qquad \text{for all } p_i \,,$$

that is, if and only if

$$\widehat{\alpha}^{\infty}_{i,0} \quad = \quad \beta_{i,0} - \frac{\beta_{-i,0}\beta_{i,-i}}{2\beta_{-i,-i}} \tag{63}$$

$$\widehat{\alpha}^{\infty}_{i} \quad = \quad \beta_{i,i} - \frac{\beta_{-i,i}\beta_{i,-i}}{2\beta_{-i,-i}} \,. \tag{64}$$

It follows from (47) and (64) that a necessary condition for the model of seller i to capture competition for the reaction function (62) is that

$$r^{\infty}_i \quad = \quad -\frac{\beta_{-i,i}}{2\beta_{-i,-i}} \,. \tag{65}$$

Substitution of expression (65) for r^{∞}_i into (56) gives

$$\beta_{i,-i} p^{\infty}_{-i} + \left(2\beta_{i,i} - \frac{\beta_{-i,i}\beta_{i,-i}}{2\beta_{-i,-i}}\right) p^{\infty}_i \quad = \quad -\beta_{i,0} \,. \tag{66}$$

Also, it follows from (48), (63), and (65) that another necessary condition is that

$$2\beta_{-i,-i} p^{\infty}_{-i} + \beta_{-i,i} p^{\infty}_i \quad = \quad -\beta_{-i,0} \,. \tag{67}$$

It follows from (66) and (67) that a necessary condition for the model of seller i to capture competition is that

$$p^{\infty}_{-i} \quad = \quad \frac{2\beta_{i,0}\beta_{-i,i}\beta_{-i,-i} - 4\beta_{-i,0}\beta_{-i,-i}\beta_{i,i} + \beta_{-i,0}\beta_{-i,i}\beta_{i,-i}}{4\beta_{-i,-i}\left(2\beta_{-i,-i}\beta_{i,i} - \beta_{-i,i}\beta_{i,-i}\right)}$$

$$p^{\infty}_{i} \quad = \quad \frac{\beta_{-i,0}\beta_{i,-i} - 2\beta_{i,0}\beta_{-i,-i}}{2\left(2\beta_{-i,-i}\beta_{i,i} - \beta_{-i,i}\beta_{i,-i}\right)} \,.$$

Note that the point $(p^{\infty}_{-1}, p^{\infty}_1)$ satisfying the necessary condition above is only one point in $\mathcal{P}$, and thus it seems a rare outcome for the model of seller i to capture competition. In addition, the necessary conditions for the models of sellers i and $-i$ to capture competition cannot be satisfied simultaneously, and thus the Market Response Hypothesis cannot hold. This is easily seen as follows: Condition (67) holds for $i = \pm 1$ if and only if

$$p_i^{\infty} \quad = \quad \frac{\beta_{-i,0}\beta_{i,-i} - 2\beta_{i,0}\beta_{-i,-i}}{4\beta_{-i,-i}\beta_{i,i} - \beta_{-i,i}\beta_{i,-i}},$$

for $i = \pm 1$, that is, if and only if the limit prices $(p_{-1}^{\infty}, p_1^{\infty})$ are the Nash equilibrium prices $(p_{-1}^{\mathrm{N}}, p_1^{\mathrm{N}})$. However, recall that $(p_{-1}^{\infty}, p_1^{\infty}) = (p_{-1}^{\mathrm{N}}, p_1^{\mathrm{N}})$ if and only if $r_i^{\infty} = 0$ for $i = \pm 1$, which, if $\beta_{-i,i} \neq 0$, contradicts the previous necessary condition that $r_i^{\infty} = -\beta_{-i,i}/(2\beta_{-i,-i})$. Thus, besides the trivial case in which $\beta_{-i,i} = 0$ for $i = \pm 1$, the Market Response Hypothesis does not hold for the reaction function (62).

5 Concluding Remarks

The analysis in this paper serves to emphasize to both practitioners and researchers that it matters whether sellers use models that do not explicitly incorporate competition and that it matters what the sellers think they know and what they think they must estimate. Here, "it matters" means that conclusions may vary widely depending on the assumptions one makes regarding these features. There are settings in which sellers' prices converge to a Nash equilibrium, but in general one should not expect that prices that arise when sellers use such flawed models will be similar to prices that arise when sellers employ models that explicitly account for competition, even though the sellers using the flawed models generate parameter estimates that appear to them to be consistent with their observed data.

Our study introduces the demand consistency property and evaluates its consequences. This property requires that each seller's estimated expected demand and actual expected demand coincide when evaluated at a pair of limiting prices. At a limit, each seller also must perceive itself to be choosing the best price based on its estimated expected demand function. The demand consistency property makes precise the notion that sellers' estimates should be calibrated to their observed data and that competition affects those estimates. However, as we have seen, this apparently reasonable notion of calibration by no means ensures that the sellers end up at the Nash equilibrium, which is typically given as the predicted outcome of competition. Moreover, convergence of prices and estimates — even when taken together with the demand consistency property — does not mean there is convergence to the Nash equilibrium.

Our results lead to the conclusion that care must be taken with the Market Response Hypothesis that monopoly models will capture competitive effects if the models are calibrated with data collected under competition. In some settings the monopoly models capture competitive effects in a limited way, and in other settings not at all.

Our results also indicate that in many cases, the sellers are better off if they use models that do not explicitly incorporate competition than they would be if they knew the expected demand as a function of prices and ended up at the Nash equilibrium. Thus, although our results do not support the Market Response Hypothesis, the results do not imply that sellers are better off using models that explicitly incorporate competition.

This paper perhaps raises as many questions as it answers. For instance, one may wonder if is it possible to identify general conditions for problems involving pricing competition, parameter estimation, and model misspecification under which there is convergence to prices that are Pareto superior to (or Pareto inferior to, or equal to) the Nash Equilibrium. A related topic for future inquiry is the extent to which sellers should attempt to improve their models. At first, this question may seem nonsensical because it would seem that "more accurate" models are better. However, as we have seen, sellers may unwittingly end up at prices that are better for them than is the Nash equilibrium. The results of this paper indicate that one must be careful when addressing this issue, and the answer will depend on the level of modeling and estimation sophistication of the sellers, as well as what the sellers think they know about each other's modeling and estimation efforts.

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Appendix

Learning and Pricing with Models that Do Not Explicitly Incorporate Competition

William L. Cooper Tito Homem-de-Mello Anton J. Kleywegt

Proposition A–1. *Suppose that $F : \mathbb{R}^d \to \mathbb{R}^d$ is a contraction mapping such that $\|F(x) - F(y)\| \leq \lambda \|x - y\|$ where $\lambda \in [0, 1)$. Consider a closed convex set $A \subseteq \mathbb{R}^d$ and let π denote projection onto A; i.e., $\pi(x) = \arg\min\{\|x - y\| : y \in A\}$. Let $x^* = (x_1^*, \ldots, x_d^*)$ denote the unique fixed point of F and suppose $x^* \in A$. Consider stochastic processes $\{X^k = (X_1^k, \ldots, X_d^k) : k = 0, 1, 2, \ldots\}$ and $\epsilon^k = \{(\epsilon_1^k, \ldots, \epsilon_d^k) : k = 1, 2, \ldots\}$ such that*

$$X^k = \pi\left(\left(1 - k^{-1}\right) X^{k-1} + k^{-1}\left(F(X^{k-1}) + \epsilon^k\right)\right) \qquad k = 1, 2, \ldots \tag{A–1}$$

Let $\mathcal{F}^k$ denote the σ-algebra generated by $X_0, \epsilon^1, \ldots, \epsilon^k$, and suppose $\mathbb{E}[\epsilon_i^k | \mathcal{F}^{k-1}] = 0$ and $\mathbb{E}[(\epsilon_i^k)^2 | \mathcal{F}^{k-1}] < M$ for $i = 1, \ldots, d$ for some constant $M < \infty$. Then $X^k \to x^*$ w.p.1.

Proof. For $x \in \mathbb{R}^d$, we will use the notation $F_i(x)$ to denote the i-th component of $F(x)$.

Let $Z^k := \left\|X^k - x^*\right\|^2 = \sum_{i=1}^d (X_i^k - x_i^*)^2$. Then

$$\begin{aligned}
Z^k &= \left\|\pi\left(\left(1 - k^{-1}\right) X^{k-1} + k^{-1}\left(F(X^{k-1}) + \epsilon^k\right)\right) - x^*\right\|^2 \\
&\leq \left\|\left(1 - k^{-1}\right) X^{k-1} + k^{-1}\left(F(X^{k-1}) + \epsilon^k\right) - x^*\right\|^2 &\text{(A–2)} \\
&= \sum_{i=1}^d \left[\left(1 - k^{-1}\right) X_i^{k-1} + k^{-1}\left(F_i(X^{k-1}) + \epsilon_i^k\right) - x_i^*\right]^2 \\
&= \sum_{i=1}^d \left[\left(1 - k^{-1}\right)\left(X_i^{k-1} - x_i^*\right) + k^{-1}\left(F_i(X^{k-1}) - x_i^* + \epsilon_i^k\right)\right]^2 &\text{(A–3)}
\end{aligned}$$

where (A–2) holds because x^* is an element of closed convex set A and π is projection onto A.

Let T_i^k denote the i-th term in the sum (A–3). We have

$$\begin{aligned}
T_i^k = {} & \left(1 - k^{-1}\right)^2 (X_i^k - x_i^*)^2 + 2(1 - k^{-1})k^{-1}(X_i^{k-1} - x_i^*)(F_i(X^{k-1}) - x_i^*) \\
& + 2(1 - k^{-1})k^{-1}(X_i^{k-1} - x_i^*)\epsilon_i^k + k^{-2}(F_i(X^{k-1}) - x_i^*)^2 \\
& + 2k^{-2}(F_i(X^{k-1}) - x_i^*)\epsilon_i^k + k^{-2}(\epsilon_i^k)^2
\end{aligned}$$

Taking conditional expectations, we obtain

$$\begin{aligned}
\mathbb{E}[T_i^k | \mathcal{F}^{k-1}] \leq {} & (1 - k^{-1})^2 (X_i^k - x_i^*)^2 + 2(1 - k^{-1})k^{-1}(X_i^{k-1} - x_i^*)(F_i(X^{k-1}) - x_i^*) \\
& + k^{-2}(F_i(X^{k-1}) - x_i^*)^2 + k^{-2}M
\end{aligned}$$

Hence by (A–3), we have

$$\mathbb{E}[Z^k|\mathcal{F}^{k-1}] \leq (1-k^{-1})^2 Z^{k-1} + 2(1-k^{-1})k^{-1}\sum_{i=1}^{d}(X_i^{k-1}-x_i^*)(F_i(X^{k-1})-F_i(x^*))$$
$$+ k^{-2}\sum_{i=1}^{d}(F_i(X^{k-1})-x_i^*)^2 + k^{-2}dM\,.$$

Next, observe that

$$\sum_{i=1}^{d}(X_i^{k-1}-x_i^*)(F_i(X^{k-1})-F_i(x^*)) \leq \left\|X^{k-1}-x^*\right\|\cdot\left\|F(X^{k-1})-F(x^*)\right\| \leq \lambda\left\|X^{k-1}-x^*\right\|^2 = \lambda Z^{k-1}$$

where the first inequality follows from the Cauchy-Schwarz inequality and the second from the fact that F is a contraction. In addition

$$\sum_{i=1}^{d}(F_i(X^{k-1})-x_i^*)^2 = \sum_{i=1}^{d}(F_i(X^{k-1})-F_i(x^*))^2 = \left\|F(X^{k-1})-F(x^*)\right\|^2 \leq \lambda^2\left\|X^{k-1}-x^*\right\|^2 = \lambda^2 Z^{k-1}$$

where the first equality holds because x^* is the fixed point of F.

So,

$$\mathbb{E}[Z^k|\mathcal{F}^{k-1}] \leq (1-k^{-1})^2 Z^{k-1} + 2(1-k^{-1})k^{-1}\lambda Z^{k-1} + k^{-2}\lambda^2 Z^{k-1} + k^{-2}dM\,.$$

Rearranging, we obtain

$$\begin{aligned}\mathbb{E}[Z^k|\mathcal{F}^{k-1}] &\leq \left\{1-2k^{-1}+k^{-2}+2k^{-1}\lambda-2k^{-2}\lambda+k^{-2}\lambda^2\right\}Z^{k-1}+k^{-2}dM\\ &\leq \left\{1-2k^{-1}(1-\lambda)+k^{-2}(1+\lambda^2)\right\}Z^{k-1}+k^{-2}dM\\ &= (1+B^{k-1})Z^{k-1}+C^{k-1}-D^{k-1}\end{aligned}$$

where

$$B^{k-1} := k^{-2}(1+\lambda^2) \qquad C^{k-1} := k^{-2}dM \qquad D^{k-1} := 2k^{-1}(1-\lambda)Z^{k-1}$$

The sequences $\{B^k\}$, $\{C^k\}$, $\{D^k\}$, and $\{Z^k\}$ are non-negative and $\sum_k B^k < \infty$ and $\sum_k C^k < \infty$ w.p.1. Hence, we conclude that from Lemma A–3 below that there exists finite random variable Z such that $Z^k \to Z$ and $\sum_k D_k < \infty$ w.p.1.

We will now argue that $Z = 0$ w.p.1, from which it follows that $X^k \to x^*$ w.p.1. Suppose for a contradiction that there exists a set of strictly positive probability upon which $Z > 0$. Then, on that set, we have $\sum_k k^{-1}Z^k = \infty$, from which it follows that $\sum_k D^k = \infty$. Hence, $P(\sum_k D_k = \infty) > 0$, which is a contraction. □

Lemma A–1. *Suppose $\beta_{i,-i} > 0$ for $i = \pm 1$ and consider the demand functions*

$$d_i(p_i, p_{-i}) \quad = \quad \beta_{i,0} + \beta_{i,i} p_i + \beta_{i,-i} p_{-i} \quad i = \pm 1. \tag{A–4}$$

Let $\theta = \sqrt{\beta_{-1,1}\beta_{1,-1}/(\beta_{-1,-1}\beta_{1,1})}$ and $\nu = \sqrt{\beta_{-1,-1}\beta_{1,-1}/(\beta_{-1,1}\beta_{1,1})}$, and consider also the demand functions

$$\tilde{d}_i(\tilde{p}_i, \tilde{p}_{-i}) \quad = \quad \tilde{\beta}_{i,0} + \tilde{\beta}_{i,i}\tilde{p}_i + \tilde{\beta}_{i,-i}\tilde{p}_{-i} \quad i = \pm 1 \tag{A–5}$$

where $\tilde{\beta}_{i,0} = \beta_{i,0}$ and $\tilde{\beta}_{i,-i} = -\theta\tilde{\beta}_{i,i}$ for $i = \pm 1$, $\tilde{\beta}_{-1,-1} = \beta_{-1,-1}$, and $\tilde{\beta}_{1,1} = \nu\beta_{1,1}$.

The demand functions (A–4) and (A–5) are equivalent in the sense that $\tilde{d}_i(\tilde{p}_i, \tilde{p}_{-i}) = d_i(p_i, p_{-i})$ for $i = \pm 1$ if $\tilde{p}_{-1} = p_{-1}$ and $\tilde{p}_1 = p_1/\nu$. Moreover, the Nash equilibrium prices $(\tilde{p}^{\mathrm{N}}_{-1}, \tilde{p}^{\mathrm{N}}_1)$ and modeling error equilibrium prices $(\tilde{p}^{\infty}_{-1}, \tilde{p}^{\infty}_1)$ for demand functions (A–5) satisfy

$$\tilde{p}^{\mathrm{N}}_{-1} \; = \; p^{\mathrm{N}}_{-1} \qquad \tilde{p}^{\mathrm{N}}_1 \; = \; p^{\mathrm{N}}_1/\nu \qquad \tilde{p}^{\infty}_{-1} \; = \; p^{\infty}_{-1} \qquad \tilde{p}^{\infty}_1 \; = \; p^{\infty}_1/\nu \tag{A–6}$$

where $(p^{\mathrm{N}}_{-1}, p^{\mathrm{N}}_1)$ and $(p^{\infty}_{-1}, p^{\infty}_1)$ are, respectively, the Nash equilibrium and modeling error equilibrium prices for demand functions (A–4).

Proof. The equivalence of (A–4) and (A–5) can be verified using simple algebra. Likewise, the relations (A–6) follow after some algebra from (6) and (22). □

Proof of Proposition 2. It follows from (6) that

$$p^{\mathrm{N}}_i \quad = \quad \frac{-\theta\beta_{-i,0}\beta_{i,i} - 2\beta_{i,0}\beta_{-i,-i}}{(4-\theta^2)\beta_{-i,-i}\beta_{i,i}}$$

and thus

$$d_i(p^{\mathrm{N}}_i, p^{\mathrm{N}}_{-i}) \quad = \quad \frac{\theta\beta_{-i,0}\beta_{i,i} + 2\beta_{i,0}\beta_{-i,-i}}{(4-\theta^2)\beta_{-i,-i}}$$

and

$$g_i(p^{\mathrm{N}}_i, p^{\mathrm{N}}_{-i}) \quad = \quad -\frac{(\theta\beta_{-i,0}\beta_{i,i} + 2\beta_{i,0}\beta_{-i,-i})^2}{(4-\theta^2)^2\beta^2_{-i,-i}\beta_{i,i}}\,.$$

Similarly, it follows from (22) that

$$p^{\infty}_i \quad = \quad \frac{-\theta\beta_{-i,0}\beta_{i,i} - \beta_{i,0}\beta_{-i,-i}}{2(1-\theta^2)\beta_{-i,-i}\beta_{i,i}}$$

and thus

$$d_i(p^{\infty}_i, p^{\infty}_{-i}) \quad = \quad \frac{\beta_{i,0}}{2}$$

and

$$g_i(p^{\infty}_i, p^{\infty}_{-i}) \quad = \quad -\frac{\theta\beta_{-i,0}\beta_{i,0}\beta_{i,i} + \beta^2_{i,0}\beta_{-i,-i}}{4(1-\theta^2)\beta_{-i,-i}\beta_{i,i}}\,.$$

Note that

$$
\begin{aligned}
&(\beta_{-1,-1},\beta_{1,1}) \in \Gamma_i^{=} \\
&\quad \Leftrightarrow\ g_i(p_i^{\mathrm{N}}, p_{-i}^{\mathrm{N}}) \ = \ g_i(p_i^{\infty}, p_{-i}^{\infty}) \\
&\quad \Leftrightarrow\ -\frac{(\theta\beta_{-i,0}\beta_{i,i} + 2\beta_{i,0}\beta_{-i,-i})^2}{(4-\theta^2)^2\beta_{-i,-i}^2\beta_{i,i}} \ = \ -\frac{\theta\beta_{-i,0}\beta_{i,0}\beta_{i,i} + \beta_{i,0}^2\beta_{-i,-i}}{4(1-\theta^2)\beta_{-i,-i}\beta_{i,i}} \\
&\quad \Leftrightarrow\ \frac{4(1-\theta^2)(\theta\beta_{-i,0}\beta_{i,i} + 2\beta_{i,0}\beta_{-i,-i})^2 - (4-\theta^2)^2\beta_{-i,-i}(\theta\beta_{-i,0}\beta_{i,0}\beta_{i,i} + \beta_{i,0}^2\beta_{-i,-i})}{4(1-\theta^2)(4-\theta^2)^2\beta_{-i,-i}^2\beta_{i,i}} \ = \ 0 \\
&\quad \Leftrightarrow\ 4(1-\theta^2)(\theta\beta_{-i,0}\beta_{i,i} + 2\beta_{i,0}\beta_{-i,-i})^2 - (4-\theta^2)^2\beta_{-i,-i}(\theta\beta_{-i,0}\beta_{i,0}\beta_{i,i} + \beta_{i,0}^2\beta_{-i,-i}) \ = \ 0 \\
&\quad \Leftrightarrow\ \theta^2\Big[\underbrace{4(1-\theta^2)\beta_{-i,0}^2}_{=:\,a}\ \beta_{i,i}^2 + \underbrace{(-(8+\theta^2)\theta\beta_{-i,0}\beta_{i,0}\beta_{-i,-i})}_{=:\ b}\ \beta_{i,i} + \underbrace{(-(8+\theta^2)\beta_{i,0}^2\beta_{-i,-i}^2)}_{=:\ c}\ \Big] \ = \ 0 \\
&\quad \Leftrightarrow\ \theta \ = \ 0 \qquad \text{or} \\
&\qquad\ \beta_{i,i} \ = \ \frac{-b \pm \sqrt{b^2-4ac}}{2a} \ = \ \frac{\beta_{i,0}}{\beta_{-i,0}}\left[\frac{(8+\theta^2)\theta \pm (4-\theta^2)\sqrt{8+\theta^2}}{8(1-\theta^2)}\right]\beta_{-i,-i}
\end{aligned}
$$

Thus, for any $(\beta_{-1,0}, \beta_{1,0}, \theta) \in (0,\infty)^2 \times (0,1)$, the set of $(\beta_{-1,-1}, \beta_{1,1})$-points such that $g_i(p_i^{\mathrm{N}}, p_{-i}^{\mathrm{N}}) = g_i(p_i^{\infty}, p_{-i}^{\infty})$ is given by two lines through the origin. Note that $(8+\theta^2)\theta + (4-\theta^2)\sqrt{8+\theta^2} > 0$, and thus one of the lines has a positive slope, and

$$
\begin{aligned}
\theta \ \in \ (0,1) \ &\Rightarrow \ \sqrt{8+\theta^2}\theta + \theta^2 \ < \ 4 \\
&\Rightarrow \ (8+\theta^2)\theta - (4-\theta^2)\sqrt{8+\theta^2} \ < \ 0
\end{aligned}
$$

and thus the other line has a negative slope. Thus the set of $(\beta_{-1,-1}, \beta_{1,1})$-points in $\mathcal{S}$ such that $g_1(p_1^{\mathrm{N}}, p_{-1}^{\mathrm{N}}) = g_1(p_1^{\infty}, p_{-1}^{\infty})$ is given by one line through the origin $\beta_{1,1} = (\beta_{1,0}/\beta_{-1,0})T(\theta)\beta_{-1,-1}$, where $T(\theta) := [(8+\theta^2)\theta + (4-\theta^2)\sqrt{8+\theta^2}]/[8(1-\theta^2)]$. Similarly, the set of $(\beta_{-1,-1}, \beta_{1,1})$-points in $\mathcal{S}$ such that $g_{-1}(p_{-1}^{\mathrm{N}}, p_1^{\mathrm{N}}) = g_{-1}(p_{-1}^{\infty}, p_1^{\infty})$ is given by one line through the origin $\beta_{-1,-1} = (\beta_{-1,0}/\beta_{1,0})T(\theta)\beta_{1,1}$, that is, $\beta_{1,1} = (\beta_{1,0}/\beta_{-1,0})[1/T(\theta)]\beta_{-1,-1}$. Note that $T(\theta) \geq \sqrt{8+\theta^2}/[2(1-\theta^2)] > \sqrt{2} > 1$ on $(0,1)$. Next we verify that $g_i(p_i^{\mathrm{N}}, p_{-i}^{\mathrm{N}}) < g_i(p_i^{\infty}, p_{-i}^{\infty})$ if $\beta_{i,i} > (\beta_{i,0}/\beta_{-i,0})T(\theta)\beta_{-i,-i}$. Specifically, consider any point $(\beta_{-1,-1}, \beta_{1,1})$ such that $\beta_{i,i} = (\beta_{i,0}/\beta_{-i,0})\beta_{-i,-i}$. Note that $\beta_{i,i} > (\beta_{i,0}/\beta_{-i,0})T(\theta)\beta_{-i,-i}$ since $T(\theta) > 1$ and $\beta_{-i,-i} < 0$.

Note that at such a point,

$$
\begin{aligned}
g_i(p_i^{\mathrm{N}},p_{-i}^{\mathrm{N}}) &< g_i(p_i^{\infty},p_{-i}^{\infty}) \\
\Leftrightarrow\; -\frac{(\theta\beta_{i,0}\beta_{-i,-i}+2\beta_{i,0}\beta_{-i,-i})^2}{(4-\theta^2)^2\beta_{-i,-i}^2\beta_{i,i}} &< -\frac{\theta\beta_{i,0}^2\beta_{-i,-i}+\beta_{i,0}^2\beta_{-i,-i}}{4(1-\theta^2)\beta_{-i,-i}\beta_{i,i}} \\
\Leftrightarrow\; \frac{(\theta+2)^2}{(4-\theta^2)^2} &< \frac{\theta+1}{4(1-\theta^2)} \\
\Leftrightarrow\; \frac{1}{(2-\theta)^2} &< \frac{1}{4(1-\theta)} \\
\Leftrightarrow\; 4-4\theta &< 4-4\theta+\theta^2\,.
\end{aligned}
$$

Thus, taking into account that $T(\theta)>1$ and $\beta_{-i,-i}<0$, it follows that

$$
\begin{aligned}
\Gamma^* &= \left\{\beta\in\mathcal{S}\;:\;\beta_{i,i}\geq\frac{\beta_{i,0}}{\beta_{-i,0}}T(\theta)\beta_{-i,-i},\; i=\pm1\right\} \\
&= \left\{\beta\in\mathcal{S}\;:\;\frac{\beta_{1,0}}{\beta_{-1,0}}T(\theta)\beta_{-1,-1}\leq\beta_{1,1}\leq\frac{\beta_{1,0}}{\beta_{-1,0}}\frac{1}{T(\theta)}\beta_{-1,-1}\right\} \;\neq\; \varnothing \\
\Gamma_i^* &= \left\{\beta\in\mathcal{S}\;:\;\beta_{i,i}\geq\frac{\beta_{i,0}}{\beta_{-i,0}}T(\theta)\beta_{-i,-i},\beta_{-i,-i}\leq\frac{\beta_{-i,0}}{\beta_{i,0}}T(\theta)\beta_{i,i}\right\} \\
&= \left\{\beta\in\mathcal{S}\;:\;\beta_{i,i}\geq\max\left\{\frac{\beta_{i,0}}{\beta_{-i,0}}T(\theta)\beta_{-i,-i},\;\frac{\beta_{i,0}}{\beta_{-i,0}}\frac{1}{T(\theta)}\beta_{-i,-i}\right\}\right\} \\
&= \left\{\beta\in\mathcal{S}\;:\;\beta_{i,i}\geq\frac{\beta_{i,0}}{\beta_{-i,0}}\frac{1}{T(\theta)}\beta_{-i,-i}\right\} \;\neq\; \varnothing \\
\Gamma^{\mathrm{N}} &= \left\{\beta\in\mathcal{S}\;:\;\beta_{i,i}\leq\frac{\beta_{i,0}}{\beta_{-i,0}}T(\theta)\beta_{-i,-i},\; i=\pm1\right\} \\
&= \left\{\beta\in\mathcal{S}\;:\;\frac{\beta_{i,0}}{\beta_{-i,0}}\frac{1}{T(\theta)}\beta_{-i,-i}\leq\beta_{i,i}\leq\frac{\beta_{i,0}}{\beta_{-i,0}}T(\theta)\beta_{-i,-i}\right\} \;=\; \varnothing\,.
\end{aligned}
$$

□

Proof of Lemma 1. We use Figure A–1 to illustrate the cases in the proof, and without loss of generality, we assume that $b_{-1}\leq b_1$.

Suppose first that (37) holds. If $x\leq -b:=(-b_{-1},-b_1)$, that is, $x\in\mathcal{R}_1$, then $\left\|\Pi(x)-\bar{\beta}^*\right\|_Q=\left\|x-\bar{\beta}^*\right\|_Q$. If $x_{-1}\leq -b_1$ and $x_1\geq -b_1$, that is, $x\in\mathcal{R}_2$, then $\Pi(x)=(x_{-1},-b_1)$. Therefore, $\Pi(x)$ is no further in Euclidean distance than x from both $\bar{\beta}^*$ and $\mathcal{D}$ [that is, $\left\|\Pi(x)-\bar{\beta}^*\right\|\leq\left\|x-\bar{\beta}^*\right\|$ and $\mathfrak{d}(\Pi(x))\leq\mathfrak{d}(x)$], and hence $\left\|\Pi(x)-\bar{\beta}^*\right\|_Q\leq\left\|x-\bar{\beta}^*\right\|_Q$ by (36). Similarly, if $x_{-1}\geq -b_{-1}$ and $x_1\leq -b_1$, that is, $x\in\mathcal{R}_3$, then $\Pi(x)=(-b_{-1},x_1)$, and $\left\|\Pi(x)-\bar{\beta}^*\right\|_Q\leq\left\|x-\bar{\beta}^*\right\|_Q$.

In the remainder of the proof we consider $x\geq(-b_1,-b_1)$, that is, $x\in\mathcal{R}_4\cup\mathcal{R}_5\cup\mathcal{R}_6$. If $x\geq(-b_{-1},-b_{-1})$, that is, $x\in\mathcal{R}_4$, then $(-b_{-1},-b_{-1})$ is no further in Euclidean distance than x from both $\bar{\beta}^*$ and $\mathcal{D}$, and hence $\left\|(-b_{-1},-b_{-1})-\bar{\beta}^*\right\|_Q\leq\left\|x-\bar{\beta}^*\right\|_Q$. Also, $\Pi(x)=\Pi(-b_{-1},-b_{-1})$,

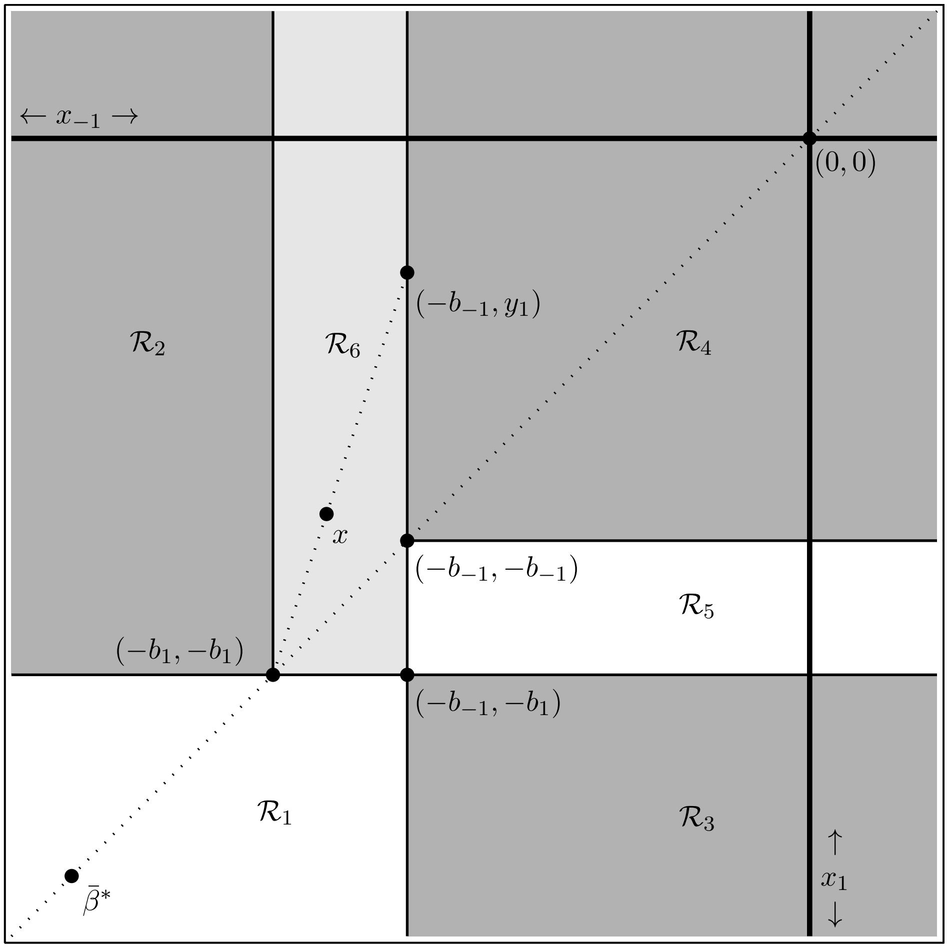

Figure A–1: Regions for x to show that $\left\|\Pi(x) - \bar{\beta}^*\right\|_Q \leq \left\|x - \bar{\beta}^*\right\|_Q$

so $\left\|\Pi(x) - \bar{\beta}^*\right\|_Q = \left\|\Pi(-b_{-1}, -b_{-1}) - \bar{\beta}^*\right\|_Q$. Therefore, (38) will hold for $x \in \mathcal{R}_4$ if (38) holds for $(-b_{-1}, -b_{-1})$.

If $x_{-1} \geq -b_{-1}$ and $-b_1 \leq x_1 < -b_{-1}$, that is, $x \in \mathcal{R}_5$, then $(-b_{-1}, x_1)$ is no further in Euclidean distance than x from both $\bar{\beta}^*$ and $\mathcal{D}$. Thus, $\left\|(-b_{-1}, x_1) - \bar{\beta}^*\right\|_Q \leq \left\|x - \bar{\beta}^*\right\|_Q$. Also, $\Pi(x) = \Pi(-b_{-1}, x_1)$, so $\left\|\Pi(x) - \bar{\beta}^*\right\|_Q = \left\|\Pi(-b_{-1}, x_1) - \bar{\beta}^*\right\|_Q$. Hence, (38) will hold for $x \in \mathcal{R}_5$ if (38) holds for $(-b_{-1}, x_1)$.

Therefore, to show that (38) holds for all $x \in \mathcal{R}_4 \cup \mathcal{R}_5 \cup \mathcal{R}_6$, it is sufficient to show that (38) holds for all points in $\mathcal{R}_6 := \{x \in \mathbb{R}^2 : -b_1 < x_{-1} \leq -b_{-1} \text{ and } x_1 \geq -b_1\}$. We begin by showing that

$$\left\|-b - \bar{\beta}^*\right\|_Q^2 \quad \leq \quad \left\|(-b_{-1}, y_1) - \bar{\beta}^*\right\|_Q^2 \tag{A–7}$$

for all $y_1 \geq -b_1$. If $y_1 = -b_1$ then (A–7) holds with equality. Suppose that $y_1 > -b_1$. It follows from (36) that

$$
\begin{aligned}
\left\|-b-\bar{\beta}^*\right\|_Q^2 &\leq \left\|(-b_{-1},y_1)-\bar{\beta}^*\right\|_Q^2 \\
\iff \quad (1-q)\left\|-b-\bar{\beta}^*\right\|^2+q(b_1-b_{-1})^2 &\leq (1-q)\left\|(-b_{-1},y_1)-\bar{\beta}^*\right\|^2+q(y_1+b_{-1})^2 \\
\iff \quad (1-q)\left[(-b_{-1}-\beta^*)^2+(-b_1-\beta^*)^2\right]+q(b_1-b_{-1})^2 & \\
\leq \quad (1-q)\left[(-b_{-1}-\beta^*)^2+(y_1-\beta^*)^2\right]+q(y_1+b_{-1})^2 & \\
\iff \quad (1-q)\left[b_1^2+2b_1\beta^*\right]+q\left[b_1^2-2b_{-1}b_1\right] &\leq (1-q)\left[y_1^2-2y_1\beta^*\right]+q\left[y_1^2+2b_{-1}y_1\right] \\
\iff \quad b_1^2-b_1\left[-2(1-q)\beta^*+2qb_{-1}\right] &\leq y_1^2+y_1\left[-2(1-q)\beta^*+2qb_{-1}\right] \\
\iff \quad (y_1+b_1)\left[(y_1-b_1)-2(1-q)\beta^*+2qb_{-1}\right] &\geq 0.
\end{aligned}
$$

Since $y_1+b_1>0$, the final inequality above holds if and only if $(y_1-b_1)-2(1-q)\beta^*+2qb_{-1}\geq 0$, i.e., if and only if $2q(\beta^*+b_{-1})\geq b_1-y_1+2\beta^*$. We are assuming that $\beta^*\leq -b_i$, $i=\pm 1$, so it follows that the inequalities above hold if and only if

$$q \quad\leq\quad \frac{b_1-y_1+2\beta^*}{2(\beta^*+b_{-1})}\,. \tag{A–8}$$

A sufficient condition for (A–8) to hold is that

$$q \quad\leq\quad \inf_{y_1>-b_1}\frac{b_1-y_1+2\beta^*}{2(\beta^*+b_{-1})} \quad=\quad \frac{2b_1+2\beta^*}{2(\beta^*+b_{-1})} \quad=\quad 1+\frac{b_1-b_{-1}}{\beta^*+b_{-1}}\,, \tag{A–9}$$

which holds by (37). Therefore, we have shown that (A–7) holds for all $y_1\geq -b_1$.

Consider any $x\in\mathcal{R}_6$ and note that x can be expressed as $x=\lambda(-b_{-1},y_1)+(1-\lambda)(-b_1,-b_1)$ for some $y_1\geq -b_1$ and $\lambda\in[0,1]$; see Figure A–1. Then

$$
\begin{aligned}
&\left\|-b-\bar{\beta}^*\right\|_Q^2-\left\|(-b_{-1},y_1)-\bar{\beta}^*\right\|_Q^2 \\
&\quad= \ (1-q)(-b_1-\beta^*)^2+q(b_1-b_{-1})^2-(1-q)(y_1-\beta^*)^2-q(y_1+b_{-1})^2.
\end{aligned} \tag{A–10}
$$

Observe also that $\Pi(x) = \lambda(-b_{-1}, -b_1) + (1-\lambda)(-b_1, -b_1)$. Therefore,

$$\begin{aligned}
&\left\|\Pi(x) - \bar{\beta}^*\right\|_Q^2 - \left\|x - \bar{\beta}^*\right\|_Q^2 \\
=\; & (1-q)\left([\lambda(-b_{-1}) + (1-\lambda)(-b_1) - \beta^*]^2 + [-b_1 - \beta^*]^2\right) + q\lambda^2(b_1 - b_{-1})^2 \\
& - (1-q)\left([\lambda(-b_{-1}) + (1-\lambda)(-b_1) - \beta^*]^2 + [\lambda(y_1) + (1-\lambda)(-b_1) - \beta^*]^2\right) - q\lambda^2(y_1 + b_{-1})^2 \\
=\; & (1-q)[\lambda(-b_1) + (1-\lambda)(-b_1) - \beta^*]^2 + q\lambda^2(b_1 - b_{-1})^2 \\
& - (1-q)[\lambda(y_1) + (1-\lambda)(-b_1) - \beta^*]^2 - q\lambda^2(y_1 + b_{-1})^2 \\
=\; & (1-q)\lambda^2(-b_1 - \beta^*)^2 + q\lambda^2(b_1 - b_{-1})^2 - (1-q)\lambda^2(y_1 - \beta^*)^2 - q\lambda^2(y_1 + b_{-1})^2 \\
& + (1-q)[(1-\lambda)^2(-b_1 - \beta^*)^2 + 2\lambda(1-\lambda)(-b_1 - \beta^*)^2] \\
& - (1-q)[(1-\lambda)^2(-b_1 - \beta^*)^2 + 2\lambda(1-\lambda)(y_1 - \beta^*)(-b_1 - \beta^*)] \\
=\; & \lambda^2\left(\left\|-b - \bar{\beta}^*\right\|_Q^2 - \left\|(-b_{-1}, y_1) - \bar{\beta}^*\right\|_Q^2\right) + 2(1-q)\lambda(1-\lambda)(-b_1 - \beta^*)(-b_1 - y_1) \\
\leq\; & 2(1-q)\lambda(1-\lambda)(-b_1 - \beta^*)(-b_1 - y_1)\,,
\end{aligned}$$

where the final equality follows from (A–10) and the inequality follows from (A–7). Hence, $\left\|\Pi(x) - \bar{\beta}^*\right\|_Q^2 \leq \left\|x - \bar{\beta}^*\right\|_Q^2$ if $(1-q)\lambda(1-\lambda)(-b_1 - \beta^*)(-b_1 - y_1) \leq 0$. The preceding inequality does indeed hold because $\beta^* \leq -b_1 \leq y_1$, $\lambda \in [0,1]$, and $q < 1$. This completes the proof in the case that (37) holds.

Next, suppose that $\beta^* + b_{\min} = 0$. Then, since $\beta^* \leq -b_{\max}$, it follows that $\beta^* = -b_{-1} = -b_1$. Therefore, regions $\mathcal{R}_5$ and $\mathcal{R}_6$ are empty. For $x \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, (38) holds by the arguments above. For $x \in \mathcal{R}_4$, $\Pi(x) = \bar{\beta}^*$ and (38) holds trivially. □

Proof of Lemma 2. Let $r = x_1/x_{-1}$. Note that

$$\begin{aligned}
h(x)^T Q y &= \begin{bmatrix} y_{-1} + \beta_d(1/r - 1) \\ y_1 + \beta_d(r-1) \end{bmatrix}^T \begin{bmatrix} 1 & -q \\ -q & 1 \end{bmatrix} \begin{bmatrix} y_{-1} \\ y_1 \end{bmatrix} \\
&= \|y\|_Q^2 + y_{-1}\big[\beta_d(1/r - 1) - q\beta_d(r-1)\big] + y_1\big[-q\beta_d(1/r-1) + \beta_d(r-1)\big] \\
&= \|y\|_Q^2 + \beta_d(r-1)\big[y_1(1 + q/r) - y_{-1}(q + 1/r)\big]\,. \qquad \text{(A–11)}
\end{aligned}$$

Next, since $r = x_1/x_{-1} = (\beta^* - y_1)/(\beta^* - y_{-1})$, it follows that $y_{-1} = y_1/r + (1 - 1/r)\,\beta^*$ and hence,

$$\begin{aligned}
&\beta_d(r-1)\big[y_1(1+q/r) - y_{-1}(q+1/r)\big] \\
&\qquad = \beta_d(r-1)\big[y_1(1+q/r-q/r-1/r^2) - \beta^*(q+1/r)(1-1/r)\big] \\
&\qquad = \frac{\beta_d(r-1)}{r^2}\big[y_1(r^2-1) - \beta^*(rq+1)(r-1)\big] \\
&\qquad = \frac{\beta_d(r-1)^2}{r^2}\big[y_1(r+1) - \beta^*(rq+1)\big] \\
&\qquad = \frac{\beta_d(r-1)^2(r+1)}{r^2}\left[(y_1-\beta^*) - \beta^*(q-1)\frac{r}{r+1}\right]. \qquad \text{(A–12)}
\end{aligned}$$

Observe that in (A–12) the first term $\beta_d(r-1)^2(r+1)/r^2$ is nonnegative. The second term is always positive, since $y_1 - \beta^* = -x_1 \geq b_1$ and

$$\beta^*(q-1)\frac{r}{r+1} \quad < \quad \beta^*(q-1) \quad \leq \quad b_{\min}.$$

It follows that $\beta_d(r-1)\big[y_1(1+q/r) - y_{-1}(q+1/r)\big] \geq 0$, and hence, it follows from (A–11) that $h^TQy \geq \|y\|_Q^2$. □

Lemma A–2. *Suppose that* $\mathbb{E}[\varepsilon_i^{k+1}|\mathcal{F}^k] = 0$, *and there is a constant* M *such that* $\mathbb{E}[(\varepsilon_i^{k+1})^2|\mathcal{F}^k] \leq M$ *for each* k *and each* i. *Then, there are positive constants* A *and* B *such that, for each* k *and each* i,

$$\mathbb{E}\left[(H^{k+1})^TQH^{k+1}\,\middle|\,\mathcal{F}^k\right] \quad \leq \quad A + B\left\|\bar{\beta}^* - \widehat{\alpha}^k\right\|_Q^2. \qquad \text{(A–13)}$$

Proof. Note that

$$\begin{aligned}
&\mathbb{E}\left[(H^{k+1})^TQH^{k+1}\,\middle|\,\mathcal{F}^k\right] \\
&= \mathbb{E}\left[(H_{-1}^{k+1})^2 + (H_1^{k+1})^2 - 2qH_{-1}^{k+1}H_1^{k+1}\,\middle|\,\mathcal{F}^k\right] \\
&= \mathbb{E}\left[\left(h_{-1}^k - 2\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\varepsilon_{-1}^{k+1}\right)^2 + \left(h_1^k - 2\frac{\widehat{\alpha}_1^k}{\beta_0}\varepsilon_1^{k+1}\right)^2 - 2q\left(h_{-1}^k - 2\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\varepsilon_{-1}^{k+1}\right)\left(h_1^k - 2\frac{\widehat{\alpha}_1^k}{\beta_0}\varepsilon_1^{k+1}\right)\,\middle|\,\mathcal{F}^k\right] \\
&= \left(h_{-1}^k\right)^2 + 4\left(\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\right)^2\mathbb{E}\left[\left(\varepsilon_{-1}^{k+1}\right)^2\,\middle|\,\mathcal{F}^k\right] - 4h_{-1}^k\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\mathbb{E}\left[\varepsilon_{-1}^{k+1}\,\middle|\,\mathcal{F}^k\right] \\
&\quad + \left(h_1^k\right)^2 + 4\left(\frac{\widehat{\alpha}_1^k}{\beta_0}\right)^2\mathbb{E}\left[\left(\varepsilon_1^{k+1}\right)^2\,\middle|\,\mathcal{F}^k\right] - 4h_1^k\frac{\widehat{\alpha}_1^k}{\beta_0}\mathbb{E}\left[\varepsilon_1^{k+1}\,\middle|\,\mathcal{F}^k\right] \\
&\quad - 2qh_{-1}^kh_1^k - 8q\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\frac{\widehat{\alpha}_1^k}{\beta_0}\mathbb{E}\left[\varepsilon_{-1}^{k+1}\varepsilon_1^{k+1}\,\middle|\,\mathcal{F}^k\right] + 4qh_{-1}^k\frac{\widehat{\alpha}_1^k}{\beta_0}\mathbb{E}\left[\varepsilon_1^{k+1}\,\middle|\,\mathcal{F}^k\right] + 4qh_1^k\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\mathbb{E}\left[\varepsilon_{-1}^{k+1}\,\middle|\,\mathcal{F}^k\right] \\
&= \left(h_{-1}^k\right)^2 + 4\left(\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\right)^2\mathbb{E}\left[\left(\varepsilon_{-1}^{k+1}\right)^2\,\middle|\,\mathcal{F}^k\right] + \left(h_1^k\right)^2 + 4\left(\frac{\widehat{\alpha}_1^k}{\beta_0}\right)^2\mathbb{E}\left[\left(\varepsilon_1^{k+1}\right)^2\,\middle|\,\mathcal{F}^k\right] \\
&\quad - 2qh_{-1}^kh_1^k - 8q\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\frac{\widehat{\alpha}_1^k}{\beta_0}\mathbb{E}\left[\varepsilon_{-1}^{k+1}\varepsilon_1^{k+1}\,\middle|\,\mathcal{F}^k\right]. \qquad \text{(A–14)}
\end{aligned}$$

Also,

$$
\begin{aligned}
\left(h_{-1}^k\right)^2 + \left(h_1^k\right)^2 - 2q h_{-1}^k h_1^k &= \left\| h^k \right\|_Q^2 \\
&= \left\| \begin{bmatrix} (\beta^* - \widehat{\alpha}_{-1}^k) + \beta_d \left(\frac{\widehat{\alpha}_{-1}^k}{\widehat{\alpha}_1^k} - 1 \right) \\ (\beta^* - \widehat{\alpha}_1^k) + \beta_d \left(\frac{\widehat{\alpha}_1^k}{\widehat{\alpha}_{-1}^k} - 1 \right) \end{bmatrix} \right\|_Q^2 \\
&= \left\| \begin{bmatrix} \left(1 - \frac{\beta_d}{\widehat{\alpha}_1^k}\right) (\beta^* - \widehat{\alpha}_{-1}^k) \\ \left(1 - \frac{\beta_d}{\widehat{\alpha}_{-1}^k}\right) (\beta^* - \widehat{\alpha}_1^k) \end{bmatrix} + \beta_d \begin{bmatrix} \frac{\beta^*}{\widehat{\alpha}_1^k} - 1 \\ \frac{\beta^*}{\widehat{\alpha}_{-1}^k} - 1 \end{bmatrix} \right\|_Q^2 \\
&\leq 2 \left\| \begin{bmatrix} \left(1 - \frac{\beta_d}{\widehat{\alpha}_1^k}\right) (\beta^* - \widehat{\alpha}_{-1}^k) \\ \left(1 - \frac{\beta_d}{\widehat{\alpha}_{-1}^k}\right) (\beta^* - \widehat{\alpha}_1^k) \end{bmatrix} \right\|_Q^2 + 2\beta_d^2 \left\| \begin{bmatrix} \frac{\beta^*}{\widehat{\alpha}_1^k} - 1 \\ \frac{\beta^*}{\widehat{\alpha}_{-1}^k} - 1 \end{bmatrix} \right\|_Q^2 \qquad \text{(A–15)}
\end{aligned}
$$

where the inequality holds because $\|x + y\|_Q^2 \leq 2\|x\|_Q^2 + 2\|y\|_Q^2$ (which is a consequence of the equality $\|x + y\|_Q^2 + \|x - y\|_Q^2 = 2\|x\|_Q^2 + 2\|y\|_Q^2$).

Next, we show that there is a finite constant C such that

$$
\left\| \begin{bmatrix} \left(1 - \frac{\beta_d}{\widehat{\alpha}_1^k}\right) (\beta^* - \widehat{\alpha}_{-1}^k) \\ \left(1 - \frac{\beta_d}{\widehat{\alpha}_{-1}^k}\right) (\beta^* - \widehat{\alpha}_1^k) \end{bmatrix} \right\|_Q^2 \leq C^2 \left\| \bar{\beta}^* - \widehat{\alpha}^k \right\|_Q^2 . \qquad \text{(A–16)}
$$

Recall that $\beta_d \geq 0$ and $\widehat{\alpha}_i^k \leq -b_i < 0$, and thus $1 \leq 1 - \beta_d/\widehat{\alpha}_i^k \leq 1 + \beta_d/b_i$. Therefore, to show (A–16), it suffices to find C such that

$$
\left\| \begin{bmatrix} a_{-1} x_{-1} \\ a_1 x_1 \end{bmatrix} \right\|_Q^2 \leq C^2 \|x\|_Q^2 \qquad \text{(A–17)}
$$

for all $x \in \mathbb{R}^2$ and all $a = (a_{-1}, a_1) \in \mathcal{S} := [1, 1 + \beta_d/b_1] \times [1, 1 + \beta_d/b_{-1}]$. To this end, consider the function

$$
f(a) := \left\| \begin{bmatrix} a_{-1} x_{-1} \\ a_1 x_1 \end{bmatrix} \right\|_Q^2 = a_{-1}^2 x_{-1}^2 + a_1^2 x_1^2 - 2q a_{-1} a_1 x_{-1} x_1
$$

Note that

$$
\nabla^2 f(a) = \begin{bmatrix} 2x_{-1}^2 & -2q x_{-1} x_1 \\ -2q x_{-1} x_1 & 2x_1^2 \end{bmatrix} .
$$

Since $2x_{-1}^2 \geq 0$, $2x_1^2 \geq 0$, and $\left|\nabla^2 f(a)\right| = 4(1 - q^2) x_{-1}^2 x_1^2 \geq 0$, it follows that $\nabla^2 f(a)$ is positive semidefinite for all a, and thus f is convex. Hence f attains its maximum over $\mathcal{S}$ at an extreme point of $\mathcal{S}$. Therefore, if (A–17) holds for all $x \in \mathbb{R}^2$ and all extreme points of $\mathcal{S}$, then (A–17) holds for all $x \in \mathbb{R}^2$ and all $a \in \mathcal{S}$.

Note that

$$\lim_{C\to\infty} \frac{\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}}{C^2 - a_{-1}a_1} \quad = \quad 1$$

Recall that $0 \leq q < 1$. Choose $C > \max\{1 + \beta_d/b_1, 1 + \beta_d/b_{-1}\}$ such that

$$q \quad \leq \quad \frac{\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}}{C^2 - a_{-1}a_1}$$

for all a in the set of four extreme points of $\mathcal{S}$.

Consider $x \in \mathbb{R}^2$ and an extreme point a of $\mathcal{S}$. If $x_{-1}x_1 > 0$ then

$$\begin{aligned}
& C^2 \|x\|_Q^2 - \left\| \begin{bmatrix} a_{-1}x_{-1} \\ a_1 x_1 \end{bmatrix} \right\|_Q^2 \\
&= (C^2 - a_{-1}^2)x_{-1}^2 + (C^2 - a_1^2)x_1^2 - 2q(C^2 - a_{-1}a_1)x_{-1}x_1 \\
&= (C^2 - a_{-1}^2)x_{-1}^2 + (C^2 - a_1^2)x_1^2 - 2\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}x_{-1}x_1 \\
&\quad + 2\left(1 - q\frac{C^2 - a_{-1}a_1}{\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}}\right)\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}x_{-1}x_1 \\
&= \left(\sqrt{C^2 - a_{-1}^2}x_{-1} - \sqrt{C^2 - a_1^2}x_1\right)^2 + 2\left(1 - q\frac{C^2 - a_{-1}a_1}{\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}}\right)\sqrt{C^2 - a_{-1}^2}\sqrt{C^2 - a_1^2}x_{-1}x_1 \\
&\geq 0
\end{aligned}$$

If $x_{-1}x_1 \leq 0$, that is, $qx_{-1}x_1 \leq 0$, then

$$\begin{aligned}
\left\| \begin{bmatrix} a_{-1}x_{-1} \\ a_1 x_1 \end{bmatrix} \right\|_Q^2 &= a_{-1}^2 x_{-1}^2 + a_1^2 x_1^2 - 2q a_{-1} a_1 x_{-1} x_1 \\
&\leq (1 + \beta_d/b_1)^2 x_{-1}^2 + (1 + \beta_d/b_{-1})^2 x_1^2 - 2q(1 + \beta_d/b_1)(1 + \beta_d/b_{-1})x_{-1}x_1 \\
&\leq C^2 x_{-1}^2 + C^2 x_1^2 - 2qC^2 x_{-1}x_1 \\
&= C^2 \|x\|_Q^2
\end{aligned}$$

Thus, (A–17) holds for all $x \in \mathbb{R}$ and all extreme points of $\mathcal{S}$. Hence, (A–16) holds.

Next, we show that there is a constant K such that

$$\left\| \begin{bmatrix} \frac{\beta^*}{\widehat{\alpha}_1^k} - 1 \\ \frac{\beta^*}{\widehat{\alpha}_{-1}^k} - 1 \end{bmatrix} \right\|_Q^2 \quad \leq \quad K\,. \tag{A–18}$$

Recall that $\beta^* < 0$ and $\widehat{\alpha}_i^k \leq -b_i < 0$, and thus $-1 < \beta^*/\widehat{\alpha}_i^k - 1 \leq -\beta^*/b_i - 1$. As before, the function $g(a) := \|a\|_Q^2$ is convex, and thus attains its maximum at an extreme point of $[-1, -\beta^*/b_1 - 1] \times [-1, -\beta^*/b_{-1} - 1]$. Consequently, (A–18) holds with

$$K \quad := \quad \max\left\{g(a) \,:\, a \in [-1, -\beta^*/b_1 - 1] \times [-1, -\beta^*/b_{-1} - 1]\right\}.$$

Therefore, it follows from (A–15), (A–16), and (A–18) that

$$\left(h_{-1}^k\right)^2 + \left(h_1^k\right)^2 - 2qh_{-1}^k h_1^k \quad \leq \quad 2C^2 \left\|\bar{\beta}^* - \widehat{\alpha}^k\right\|_Q^2 + 2\beta_d^2 K. \tag{A–19}$$

Next, observe that $q\widehat{\alpha}_{-1}^k \widehat{\alpha}_1^k \geq 0$, and $\varepsilon_{-1}^{k+1}\varepsilon_1^{k+1} \geq -[(\varepsilon_{-1}^{k+1})^2 + (\varepsilon_1^{k+1})^2]/2$, and thus

$$-q\widehat{\alpha}_{-1}^k \widehat{\alpha}_1^k \mathbb{E}\left[\varepsilon_{-1}^{k+1}\varepsilon_1^{k+1} \,\middle|\, \mathcal{F}^k\right] \quad \leq \quad q\widehat{\alpha}_{-1}^k \widehat{\alpha}_1^k \mathbb{E}\left[(\varepsilon_{-1}^{k+1})^2 + (\varepsilon_1^{k+1})^2]/2 \,\middle|\, \mathcal{F}^k\right] \quad \leq \quad q\widehat{\alpha}_{-1}^k \widehat{\alpha}_1^k M.$$

Hence,

$$\begin{aligned} 4\left(\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon_{-1}^{k+1}\right)^2 \,\middle|\, \mathcal{F}^k\right] + 4\left(\frac{\widehat{\alpha}_1^k}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon_1^{k+1}\right)^2 \,\middle|\, \mathcal{F}^k\right] - 8q\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\frac{\widehat{\alpha}_1^k}{\beta_0}\mathbb{E}\left[\varepsilon_{-1}^{k+1}\varepsilon_1^{k+1} \,\middle|\, \mathcal{F}^k\right] \\ \leq \quad \frac{4}{\beta_0^2} M \left[\left(\widehat{\alpha}_{-1}^k\right)^2 + \left(\widehat{\alpha}_1^k\right)^2 + 2q\widehat{\alpha}_{-1}^k\widehat{\alpha}_1^k\right]. \end{aligned} \tag{A–20}$$

Next we show how to choose $\tilde{C} > 1$ so that

$$\tilde{C}^2 \|x\|_Q^2 \quad \geq \quad x_{-1}^2 + x_1^2 + 2qx_{-1}x_1 \tag{A–21}$$

for all $x \in \mathbb{R}_-^2$. Note that

$$\lim_{\tilde{C}\to\infty} \frac{\tilde{C}^2 - 1}{\tilde{C}^2 + 1} \quad = \quad 1$$

Recall that $0 \leq q < 1$. Choose $\tilde{C} > 1$ such that

$$q \quad \leq \quad \frac{\tilde{C}^2 - 1}{\tilde{C}^2 + 1}$$

Then

$$\begin{aligned} & \tilde{C}^2 \|x\|_Q^2 - \left[x_{-1}^2 + x_1^2 + 2qx_{-1}x_1\right] \\ & = \quad (\tilde{C}^2 - 1)x_{-1}^2 + (\tilde{C}^2 - 1)x_1^2 - 2q(\tilde{C}^2 + 1)x_{-1}x_1 \\ & = \quad (\tilde{C}^2 - 1)x_{-1}^2 + (\tilde{C}^2 - 1)x_1^2 - 2\sqrt{\tilde{C}^2 - 1}\sqrt{\tilde{C}^2 - 1}x_{-1}x_1 \\ & \qquad + 2\left(1 - q\frac{\tilde{C}^2 + 1}{\tilde{C}^2 - 1}\right)\sqrt{\tilde{C}^2 - 1}\sqrt{\tilde{C}^2 - 1}x_{-1}x_1 \\ & = \quad \left(\sqrt{\tilde{C}^2 - 1}x_{-1} - \sqrt{\tilde{C}^2 - 1}x_1\right)^2 + 2\left(1 - q\frac{\tilde{C}^2 + 1}{\tilde{C}^2 - 1}\right)\sqrt{\tilde{C}^2 - 1}\sqrt{\tilde{C}^2 - 1}x_{-1}x_1 \quad \geq \quad 0. \end{aligned}$$

Thus, it follows from (A–20) and (A–21) that

$$
\begin{aligned}
4\left(\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon_{-1}^{k+1}\right)^2 \,\middle|\, \mathcal{F}^k\right] + 4\left(\frac{\widehat{\alpha}_1^k}{\beta_0}\right)^2 \mathbb{E}\left[\left(\varepsilon_1^{k+1}\right)^2 \,\middle|\, \mathcal{F}^k\right] - 8q\frac{\widehat{\alpha}_{-1}^k}{\beta_0}\frac{\widehat{\alpha}_1^k}{\beta_0}\mathbb{E}\left[\varepsilon_{-1}^{k+1}\varepsilon_1^{k+1} \,\middle|\, \mathcal{F}^k\right] & \\
\leq \frac{4}{\beta_0^2} M\tilde{C}^2 \left\|\widehat{\alpha}^k\right\|_S^2 & \\
= \frac{4M\tilde{C}^2}{\beta_0^2}\left\|\widehat{\alpha}^k - \bar{\beta}^* + \bar{\beta}^*\right\|_Q^2 & \\
\leq \frac{8M\tilde{C}^2}{\beta_0^2}\left\|\widehat{\alpha}^k - \bar{\beta}^*\right\|_Q^2 + \frac{8M\tilde{C}^2}{\beta_0^2}\left\|\bar{\beta}^*\right\|_Q^2 . & \qquad \text{(A–22)}
\end{aligned}
$$

The result follows from (A–14), (A–19), and (A–22) by choosing

$$
\begin{aligned}
A &:= 2\beta_d^2 K + \frac{8M\tilde{C}^2}{\beta_0^2}\left\|\bar{\beta}^*\right\|_Q^2 \quad > \quad 0 \\
B &:= 2C^2 + \frac{8M\tilde{C}^2}{\beta_0^2} \quad > \quad 0.
\end{aligned}
$$

□

We will use Lemma A–3, which was established by Robbins and Siegmund (1971).

Lemma A–3. *Consider a sequence of finite, nonnegative random variables B^k, C^k, D^k, Z^k, adapted to the σ-field $\mathcal{F}^k$, that satisfy*

$$
\mathbb{E}\left[Z^{k+1} \,|\, \mathcal{F}^k\right] \quad \leq \quad (1 + B^k)Z^k + C^k - D^k.
$$

Then, there is a finite random variable Z such that, on the set $\{\sum_k B^k < \infty, \sum_k C^k < \infty\}$, w.p.1,

$$
\lim_{k\to\infty} Z^k \quad = \quad Z \qquad \text{and} \qquad \sum_k D^k \quad < \quad \infty.
$$

Lemma A–4. *If (55) holds, then the system (56) is nonsingular.*

Proof. Recall (56):

$$
\beta_{i,-i}p_{-i}^* + (\beta_{i,-i}r_i^* + 2\beta_{i,i})p_i^* \quad = \quad -\beta_{i,0}\,, \qquad i = \pm 1.
$$

Thus, the system (56) is nonsingular if and only if the determinant

$$
\Delta \quad := \quad 4\,\beta_{-1,-1}\,\beta_{1,1} + 2\,\beta_{-1,-1}\,\beta_{1,-1}\,r_1^* + 2\,\beta_{1,1}\,\beta_{-1,1}\,r_{-1}^* - \beta_{-1,1}\,\beta_{1,-1}\,(1 - r_{-1}^* r_1^*) \quad \neq \quad 0
$$

Next we show that if (55) holds, then $\Delta > 0$. Note that $r^*_{-i} \le -\beta_{-i,-i}/\beta_{-i,i}$, $\beta_{-i,i} \ge 0$, and $\beta_{-i,-i} < 0$ imply that $\beta_{-i,i} r^*_{-i} + 2\beta_{-i,-i} < 0$. Thus

$$\begin{aligned}
\Delta &= \left(\beta_{-1,1} r^*_{-1} + 2\beta_{-1,-1}\right) \beta_{1,-1} r^*_1 + 2\left(\beta_{-1,1} r^*_{-1} + 2\beta_{-1,-1}\right) \beta_{1,1} - \beta_{-1,1}\beta_{1,-1} \\
&\ge -\left(\beta_{-1,1} r^*_{-1} + 2\beta_{-1,-1}\right) \beta_{1,1} + 2\left(\beta_{-1,1} r^*_{-1} + 2\beta_{-1,-1}\right) \beta_{1,1} - \beta_{-1,1}\beta_{1,-1} \\
&= \beta_{1,1}\beta_{-1,1} r^*_{-1} + 2\beta_{-1,-1}\beta_{1,1} - \beta_{-1,1}\beta_{1,-1} \\
&\ge -\beta_{-1,-1}\beta_{1,1} + 2\beta_{-1,-1}\beta_{1,1} - \beta_{-1,1}\beta_{1,-1} \\
&= \beta_{-1,-1}\beta_{1,1} - \beta_{-1,1}\beta_{1,-1} \quad > \quad 0
\end{aligned}$$

where the first inequality follows from $\beta_{-1,1} r^*_{-1} + 2\beta_{-1,-1} < 0$, $\beta_{1,-1} \ge 0$, and (55); the second inequality follows from $\beta_{1,1} < 0$, $\beta_{-1,1} \ge 0$, and (55); and the third inequality follows from (3). □

Lemma A–5. *$p_i(r_{-1}, r_1) > 0$ for all $(r_{-1}, r_1) \in \mathcal{R}$*

Proof. Recall that

$$p_i(r_{-1}, r_1) \quad := \quad \frac{-2\,\beta_{i,0}\,\beta_{-i,-i} + \beta_{-i,0}\,\beta_{i,-i} - \beta_{i,0}\,\beta_{-i,i}\,r_{-i}}{4\,\beta_{-i,-i}\,\beta_{i,i} + 2\,\beta_{-i,-i}\,\beta_{i,-i}\,r_i + 2\,\beta_{i,i}\,\beta_{-i,i}\,r_{-i} - \beta_{-i,i}\,\beta_{i,-i}\,(1 - r_{-i}\,r_i)}.$$

Note that the denominator on the right side is the determinant Δ, and it was shown above in the proof of Lemma A–4 that if $r_i \le -\beta_{i,i}/\beta_{i,-i}$ for $i = \pm 1$, then $\Delta > 0$. Next, consider the numerator on the right side:

$$\begin{aligned}
-2\,\beta_{i,0}\,\beta_{-i,-i} + \beta_{-i,0}\,\beta_{i,-i} - \beta_{i,0}\,\beta_{-i,i}\,r_{-i} &\ge -2\,\beta_{i,0}\,\beta_{-i,-i} + \beta_{-i,0}\,\beta_{i,-i} + \beta_{i,0}\,\beta_{-i,-i} \\
&= -\beta_{i,0}\,\beta_{-i,-i} + \beta_{-i,0}\,\beta_{i,-i} \quad > \quad 0
\end{aligned}$$

where the first inequality follows because $r_i \le -\beta_{i,i}/\beta_{i,-i}$ for $i = \pm 1$ for $(r_{-1}, r_1) \in \mathcal{R}$ and the second inequality follows from $\beta_{i,0} > 0$, $\beta_{i,i} < 0$, and $\beta_{i,-i} \ge 0$ for $i = \pm 1$. □

Proof of Lemma 3. Fix $i = \pm 1$. By writing r_i^{k+1} as in (46) and using recursive expressions for the averages, we obtain the following equivalences:

$$
\begin{aligned}
& r_i^k = \frac{\overline{pp}^{k+1} - \overline{p}_{-i}^{k+1}\overline{p}_i^{k+1}}{\overline{p^2}_i^{k+1} - \left(\overline{p}_i^{k+1}\right)^2} \quad \left(= r_i^{k+1} \right) \\
\Leftrightarrow \quad & \overline{pp}^{k+1} - \overline{p}_{-i}^{k+1}\overline{p}_i^{k+1} = r_i^k \left[\overline{p^2}_i^{k+1} - \left(\overline{p}_i^{k+1}\right)^2 \right] \\
\Leftrightarrow \quad & \frac{k}{k+1}\overline{pp}^k + \frac{1}{k+1}p_{-i}^k p_i^k - \left(\frac{k}{k+1}\overline{p}_{-i}^k + \frac{1}{k+1}p_{-i}^k \right) \left(\frac{k}{k+1}\overline{p}_i^k + \frac{1}{k+1}p_i^k \right) \\
& = r_i^k \left[\frac{k}{k+1}\overline{p^2}_i^k + \frac{1}{k+1}\left(p_i^k\right)^2 - \left(\frac{k}{k+1}\overline{p}_i^k + \frac{1}{k+1}p_i^k \right)^2 \right] \\
\Leftrightarrow \quad & \frac{k}{k+1}\left(\overline{pp}^k - \overline{p}_{-i}^k\overline{p}_i^k \right) + \frac{k}{k+1}\overline{p}_{-i}^k\overline{p}_i^k - \left(\frac{k}{k+1}\right)^2 \overline{p}_{-i}^k\overline{p}_i^k + \frac{1}{k+1}p_{-i}^k p_i^k - \left(\frac{1}{k+1}\right)^2 p_{-i}^k p_i^k \\
& -\frac{k}{(k+1)^2}\overline{p}_{-i}^k p_i^k - \frac{k}{(k+1)^2}\overline{p}_i^k p_{-i}^k \\
& = r_i^k \left[\frac{k}{k+1}\left(\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2 \right) + \frac{k}{k+1}\left(\overline{p}_i^k\right)^2 - \left(\frac{k}{k+1}\right)^2 \left(\overline{p}_i^k\right)^2 + \frac{1}{k+1}\left(p_i^k\right)^2 - \left(\frac{1}{k+1}\right)^2 \left(p_i^k\right)^2 \right. \\
& \left. -\frac{2k}{(k+1)^2}\overline{p}_i^k p_i^k \right] \\
\Leftrightarrow \quad & \frac{k}{(k+1)^2}\overline{p}_{-i}^k\overline{p}_i^k + \frac{k}{(k+1)^2}p_{-i}^k p_i^k - \frac{k}{(k+1)^2}\left(\overline{p}_{-i}^k p_i^k + \overline{p}_i^k p_{-i}^k \right) + \frac{k}{k+1}r_i^k \left(\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2 \right) \\
& = r_i^k \left[\frac{k}{(k+1)^2}\left(\overline{p}_i^k\right)^2 + \frac{k}{(k+1)^2}\left(p_i^k\right)^2 - \frac{2k}{(k+1)^2}\overline{p}_i^k p_i^k \right] + \frac{k}{k+1}r_i^k \left(\overline{p^2}_i^k - \left(\overline{p}_i^k\right)^2 \right) \\
\Leftrightarrow \quad & \left(\overline{p}_{-i}^k - p_{-i}^k \right)\left(\overline{p}_i^k - p_i^k \right) = r_i^k \left(\overline{p}_i^k - p_i^k \right)^2 .
\end{aligned}
$$

It follows that the ratio r_i^k defined in (46) satisfies $r_i^{k_0+1} = r_i^{k_0}$ if and only if either (58) or (59) holds. In that case, it follows from (47) that $\widehat{\alpha}_i^{k_0+1} = \widehat{\alpha}_i^{k_0}$.

Next we verify that $\widehat{\alpha}_{i,0}^{k_0+1} = \widehat{\alpha}_{i,0}^{k_0}$ if (58) holds for both $i = \pm 1$ or if (59) holds. In either case, $r_i^{k_0+1} = r_i^{k_0}$ by the above argument, and hence from (48) we have the following equivalences:

$$
\begin{aligned}
& \widehat{\alpha}_{i,0}^{k_0+1} = \widehat{\alpha}_{i,0}^{k_0} \\
\Leftrightarrow \quad & \overline{p}_{-i}^{k_0+1} - \overline{p}_i^{k_0+1} r_i^{k_0+1} = \overline{p}_{-i}^{k_0} - \overline{p}_i^{k_0} r_i^{k_0} \\
\Leftrightarrow \quad & \left(\frac{k_0}{k_0+1}\overline{p}_{-i}^{k_0} + \frac{1}{k_0+1}p_{-i}^{k_0} \right) - \left(\frac{k_0}{k_0+1}\overline{p}_i^{k_0} + \frac{1}{k_0+1}p_i^{k_0} \right) r_i^{k_0} = \overline{p}_{-i}^{k_0} - \overline{p}_i^{k_0} r_i^{k_0} \\
\Leftrightarrow \quad & \frac{1}{k_0+1}\left(\overline{p}_{-i}^{k_0} - p_{-i}^{k_0} \right) = \frac{1}{k_0+1}\left(\overline{p}_i^{k_0} - p_i^{k_0} \right) r_i^{k_0} \\
\Leftrightarrow \quad & \overline{p}_{-i}^{k_0} - p_{-i}^{k_0} = r_i^{k_0}\left(\overline{p}_i^{k_0} - p_i^{k_0} \right) . \qquad \text{(A–23)}
\end{aligned}
$$

The statement (A–23) is true if (58) holds for both $i = \pm 1$ or if (59) holds.

It follows from the above developments that if (58) holds for both $i = \pm 1$ or (59) holds for both $i = \pm 1$, then $r_i^{k_0+1} = r_i^{k_0}$, $\widehat{\alpha}_i^{k_0+1} = \widehat{\alpha}_i^{k_0}$, and $\widehat{\alpha}_{i,0}^{k_0+1} = \widehat{\alpha}_{i,0}^{k_0}$ for both $i = \pm 1$. Consequently, $p_i^{k_0+1} = p_i^{k_0}$ for both $i = \pm 1$ by (49). It follows that

$$\overline{p}_i^{k_0+1} - p_i^{k_0+1} \quad = \quad \frac{k_0}{k_0+1}\overline{p}_i^{k_0} + \frac{1}{k_0+1}p_i^{k_0} - p_i^{k_0} \quad = \quad \frac{k_0}{k_0+1}\left(\overline{p}_i^{k_0} - p_i^{k_0}\right) \qquad \text{for } i = \pm 1,$$

and thus (58) holds for both $i = \pm 1$ or (59) holds for both $i = \pm 1$ each with $k_0 + 1$ in place of k_0. This implies $r_i^{k_0+2} = r_i^{k_0+1}$ and $p_i^{k_0+2} = p_i^{k_0+1}$ for $i = \pm 1$. Repeating the argument, it follows that $r_i^{k+1} = r_i^k$ and $p_i^{k+1} = p_i^k$ for $i = \pm 1$ for all $k \geq k_0$ and therefore $r_i^k = r_i^{k_0}$ and $p_i^k = p_i^{k_0}$ for $i = \pm 1$ for all $k \geq k_0$; that is, the process is stationary at period k_0.

Conversely, suppose the process is stationary at period k_0. Then $r_i^{k_0+1} = r_i^{k_0}$ for $i = \pm 1$ and, as seen above, it must hold that $\widehat{\alpha}_i^{k_0+1} = \widehat{\alpha}_i^{k_0}$ for $i = \pm 1$. Together with the fact that $p_i^{k_0+1} = p_i^{k_0}$ for $i = \pm 1$, this implies by (49) that $\widehat{\alpha}_i^{k_0+1} = \widehat{\alpha}_i^{k_0}$ for $i = \pm 1$, and hence it follows from (A–23) that $\overline{p}_{-i}^{k_0} - p_{-i}^{k_0} = r_i^{k_0}\left(\overline{p}_i^{k_0} - p_i^{k_0}\right)$ for $i = \pm 1$. This implies that either (58) holds for both $i = \pm 1$ or else (59) holds for both $i = \pm 1$.

To show the second assertion of the lemma, consider the linear system

$$\begin{aligned} \overline{p}_{-1}^{k_0} - p_{-1}^{k_0} &= r_1^{k_0}\left(\overline{p}_1^{k_0} - p_1^{k_0}\right) \\ \overline{p}_1^{k_0} - p_1^{k_0} &= r_{-1}^{k_0}\left(\overline{p}_{-1}^{k_0} - p_{-1}^{k_0}\right). \end{aligned}$$

It is easy to see that the determinant of this system is $1 - r_1^{k_0} r_{-1}^{k_0}$. It follows that if $r_1^{k_0} r_{-1}^{k_0} < 1$, then the system has a unique solution, given by $\overline{p}_i^{k_0} = p_i^{k_0}$, $i = \pm 1$. □

Proof of Proposition 4. Consider $(p_{-1}^*, p_1^*) \in \mathcal{P}'$. By the definition of $\mathcal{P}'$, there exists a pair $(r_{-1}^*, r_1^*) \in \mathcal{R}'$ such that $p_i^* = p_i(r_{-1}^*, r_1^*)$ for $i = \pm 1$. Suppose initially that $0 < r_{-1}^* r_1^* < 1$. In what follows, the superscript 3 refers to $k = 3$, i.e. the values calculated from the three initial price pairs p^0, p^1, p^2.

To simplify the notation, define

$$\begin{aligned} &w := \overline{pp}^3 \qquad s := \overline{p}_{-1}^3 \qquad t := \overline{p}_1^3 \\ &\qquad\qquad\quad\; u := \overline{p^2}_{-1}^3 \quad\; v := \overline{p^2}_1^3 \end{aligned}$$

and

$$\begin{aligned} a &:= p_{-1}^0 \qquad c := p_{-1}^1 \qquad x := p_{-1}^2 \\ b &:= p_1^0 \qquad\;\; d := p_1^1 \qquad\;\; y := p_1^2. \end{aligned}$$

Then the following relations hold:

$$ab + cd + xy = 3w \tag{A–24}$$

$$a + c \ \ + x \ \ = 3s \tag{A–25}$$

$$b + d \ \ + y \ \ = 3t \tag{A–26}$$

$$a^2 + c^2 + x^2 = 3u \tag{A–27}$$

$$b^2 + d^2 + y^2 = 3v. \tag{A–28}$$

Our goal is to determine initial points (i.e., a, b, c, d, x and y) such that the stationary condition $\overline{p}_i^k = p_i^k$ holds for $k = 3$ and also $r_i^3 = r_i^*$, $i = \pm 1$ — which then will automatically imply that $p_i^k = p_i^*$. It is easy to see from (49) that, if $r_i^3 = r_i^*$ holds, then the condition $\overline{p}_i^3 = p_i^3$ becomes equivalent to imposing directly that $\overline{p}_i^3 = p_i^*$, where p_i^* is given by (57). That is, we want the initial points a, b, c, d, x and y to satisfy

$$\frac{w - st}{u - s^2} = r_{-1}^* \tag{A–29}$$

$$\frac{w - st}{v - t^2} = r_1^* \tag{A–30}$$

$$s = p_{-1}^* \tag{A–31}$$

$$t = p_1^*, \tag{A–32}$$

where w, s, t, u and v are functions of a, b, c, d, x and y (cf. (A–24)-(A–28)). Thus, the above system has four equations and six unknowns, so there are two degrees of freedom. By fixing $y = 0$, we can obtain a, b, c, d as a function of x as follows:

$$\begin{aligned}
b &= \frac{3p_1^*}{2} \pm (x - p_{-1}^* + r_1^* p_1^*) \sqrt{\frac{3r_{-1}^*}{4r_1^*(1 - r_{-1}^* r_1^*)}} \\
a &= -\frac{1}{2} \left[3\, r_{-1}^*\, x^2 + 3\, r_{-1}^*\, (p_{-1}^*)^2 - 6\, r_{-1}^*\, p_{-1}^*\, x + 3\, r_1^*\, (p_1^*)^2 \right. \\
&\quad + 6\, r_{-1}^*\, r_1^*\, p_1^*\, p_{-1}^* - 6\, p_{-1}^*\, b\, r_{-1}^*\, r_1^* - 2\, x\, b + 6\, p_{-1}^*\, b + 2\, x\, b\, r_{-1}^*\, r_1^* \\
&\quad \left. + 6\, x\, p_1^* - 12\, p_{-1}^*\, p_1^* \right] / \left[(-1 + r_{-1}^*\, r_1^*)\, (2\, b - 3\, p_1^*) \right] \\
c &= -\frac{1}{2} \left[6\, p_{-1}^*\, b - 6\, p_{-1}^*\, p_1^* - 6\, p_{-1}^*\, b\, r_{-1}^*\, r_1^* + 12\, r_{-1}^*\, r_1^*\, p_1^*\, p_{-1}^* \right. \\
&\quad - 2\, x\, b + 2\, x\, b\, r_{-1}^*\, r_1^* - 6\, r_{-1}^*\, r_1^*\, p_1^*\, x - 3\, r_{-1}^*\, x^2 - 3\, r_{-1}^*\, (p_{-1}^*)^2 \\
&\quad \left. + 6\, r_{-1}^*\, p_{-1}^*\, x - 3\, r_1^*\, (p_1^*)^2 \right] / \left[(-1 + r_{-1}^*\, r_1^*)\, (2\, b - 3\, p_1^*) \right] \\
d &= 3\, p_1^* - b.
\end{aligned}$$

It remains to show that it is possible to choose x such that (i) the denominators of the expressions for a and c above do not vanish, and (ii) the denominators of (A–29) and (A–30) do not vanish. To show (i), notice that b is a linear function of x; hence, there is only one value of x for which $2b - 3p_1^* = 0$. Since $r_{-1}^* r_1^* < 1$, those denominators are non-zero for all but one value of x. To show (ii), note that, as observed earlier, the denominator of (A–29) vanishes if and only if $p_{-1}^0 = p_{-1}^1 = p_{-1}^2$ (and similarly for (A–30)). It follows from the above expressions for a and c that

$$c = a \quad \Leftrightarrow \quad \text{either } x = p_{-1}^* - p_1^* r_1^* \text{ or } x = p_{-1}^* - p_1^*/r_{-1}^*$$

so we can always choose x such that $c \neq a$, which guarantees that $u - s^2 > 0$. Finally, by choosing x such that $3p_1^* - b \neq 0$, we have that $d \neq 0 = y$, which guarantees that $v - t^2 > 0$.

Next, consider the case $r_1^* = r_{-1}^* = 0$. It is easy to see that the system (A–29)-(A–32) has three equations and six unknowns (provided that the denominators in (A–29) and (A–30) are nonzero), so there are three degrees of freedom. By fixing two of the variables to be zero (say, x and c), we can find values for the remaining variables that ensure that $u - s^2 \neq 0$ and $v - t^2 \neq 0$. One such set of values is

$$\begin{aligned} a &= 3p_{-1}^* \\ b &= p_1^* \\ d &= 0 \\ y &= 2p_1^*. \end{aligned}$$

Finally, we consider the case $r_1^* r_{-1}^* = 1$. We will use only two initial data points p^0, p^1, so w, s, t, u and v are re-defined accordingly with 3 replaced by 2. Relations (A–24)-(A–28) become

$$ab + cd = 2w \tag{A–33}$$

$$a + c = 2s \tag{A–34}$$

$$b + d = 2t \tag{A–35}$$

$$a^2 + c^2 = 2u \tag{A–36}$$

$$b^2 + d^2 = 2v. \tag{A–37}$$

As before, we want to impose the conditions $r_i^2 = r_i^*$ and $\overline{p}_i^2 = p_i^*$, that is, (A–29)-(A–32). However, in this case the system is much simpler, since it follows from (A–33)–(A–37) that

$$r_{-1}^* = \frac{d - b}{c - a}, \qquad r_1^* = \frac{c - a}{d - b}.$$

The second equation is redundant, since by assumption $r^*_{-1} r^*_1 = 1$. Together with (A–31) and (A–32), we then have three equations and four unknowns. By fixing $b = 0$ it follows that

$$
\begin{aligned}
a &= p^*_{-1} - r^*_1 p^*_1 \\
c &= p^*_{-1} + r^*_1 p^*_1 \\
d &= 2p^*_1.
\end{aligned}
$$

We conclude by noticing that condition (61) ensures that $\widehat{\alpha}^3_i < 0$, cf. (47). Therefore, the same values are repeated all $k = 4, 5, \ldots$, so the system becomes stationary. □

References

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