Sum of Squares Hierarchies for the Stability Number of a Graph

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Problem

The *stability number* $\alpha(G)$ of a graph $G=(V,E)$ is the largest cardinality of a stable set in G. Computing $\alpha(G)$ is a central problem in combinatorial optimization, well-known to be NP-hard [Karp, 1972].

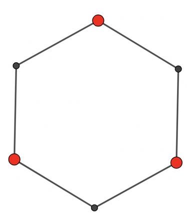


Figure 1: $\alpha=3$
ϑ-rank$(G)=0$
G acritical

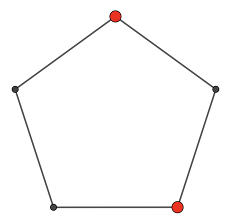


Figure 2: $\alpha=2$
ϑ-rank$(G)=1$
G critical

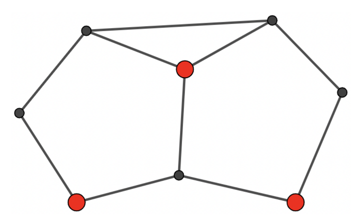


Figure 3: $\alpha=3$
ϑ-rank$(G)=2$
G critical.

A starting point to define hierarchies of approximations for the stability number is the following formulation by Motzkin and Straus [1965], which expresses $\alpha(G)$ via quadratic optimization over the standard simplex $\Delta_n=\{x\in\mathbb{R}^n : x\geq 0, \sum_{i=1}^n x_i=1\}$:

$$\frac{1}{\alpha(G)}=\min\{x^T(A_G+I)x : x\in\Delta_n\}, \qquad \text{(M-S)}$$

where A_G is the adjacency matrix of G.

Based on (M-S), de Klerk and Pasechnik in [2] proposed the copositive reformulation:

$$\alpha(G)=\min\{t : t(I+A_G)-J\in \text{COP}_n\},$$

where $\text{COP}_n=\{M\in\mathcal{S}^n : x^TMx\geq 0\ \forall x\in\mathbb{R}^n_+\}$ is the copositive cone. Parrilo [1] introduced the cones:

$$\mathcal{K}_n^{(r)}=\Big\{M\in\mathcal{S}^n : \Big(\sum_{i=1}^n x_i^2\Big)^r (x^{\circ 2})^T M x^{\circ 2} \text{ is a sum of squares}\Big\}.$$

Notice that $\mathcal{K}_n^{(r)}\subseteq \text{COP}_n$ for any $r\geq 0$. Here, $x^{\circ 2}=(x_1^2,x_2^2,\dots,x_n^2)$. De Klerk and Pasechnik [2] used these cones to define the following parameters:

$$\vartheta^{(r)}(G)=\min\{t : t(I+A_G)-J\in\mathcal{K}_n^{(r)}\},$$

Some known results about this hierarchy are the following:

- $\alpha(G)\leq\vartheta^{(r+1)}(G)\leq\vartheta^{(r)}(G)$.
- $\vartheta^{(r)}(G)\to\alpha(G)$ as $r\to\infty$.
- $\vartheta^{(0)}=\vartheta'(G)$. Here, $\vartheta'(G)$ is the stengthening of the Lovász theta number (with nonnegativity).
- $\vartheta^{(r)}(G)<\alpha(G)+1$ for $r\geq\alpha(G)^2$ (see [2]).
- $\vartheta^{(\alpha(G)-1)}(G)=\alpha(G)$ for every graph with $\alpha(G)\leq 8$ (see [3]).

Conjecture 1 (De Klerk and Pasechnik, 2002). *For any graph G we have: $\vartheta^{(\alpha(G)-1)}(G)=\alpha(G)$.*

Is it not even know whether finite convergence holds:

Conjecture 2 (weaker). *For any graph G there exists $r\in\mathbb{N}$ such that $\vartheta^{(r)}(G)=\alpha(G)$.*

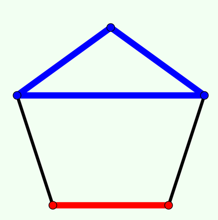
In other words Conjecture 2 is claiming that the polynomial

$$\Big(\sum_{i=1}^n x_i^2\Big)^r (x^{\circ 2})^T\Big(\alpha(G)(A_G+I)-J\Big)x^{\circ 2} \qquad (1)$$

is a sum of squares for some $r\in\mathbb{N}$, while Conjecture 1 is claiming the same result for $r=\alpha(G)-1$. Define the ϑ-rank(G) as the smallest r for which the polynomial (1) is sum of squares or, equivalently, the smallest r for which $\vartheta^{(r)}(G)=\alpha(G)$.

Example 1

If $\overline{\chi}(G)=\alpha(G)$ (that is, V is covered by $\alpha(G)$ cliques), then ϑ-rank$(G)=0$.



Example 2

Let $G=C_5$ be the 5-cycle and let $M=2(A_G+I)-J$. then

$$\Big(\sum_{i=1}^5 x_i^2\Big)x^{\circ 2T}Mx^{\circ 2}=\sum_{circ}x_1^2(x_5^2+x_1^2+x_2^2-x_3^2-x_4^2)^2$$
$$+4(x_1^2x_2^2x_4^2+x_2^2x_3^2x_5^2+x_3^2x_4^2x_1^2)$$
$$+4(x_4^2x_5^2x_2^2+x_5^2x_1^2x_3^2).$$

Hence, it is a sum of squares. It shows that ϑ-rank$(C_5)\leq 1$.

Role of Critical Edges

An edge e of a graph G is *critical* if $\alpha(G\setminus e)=\alpha(G)+1$. We say that G is *critical* if all its edges are critical and *acritical* if it does not have critical edges.

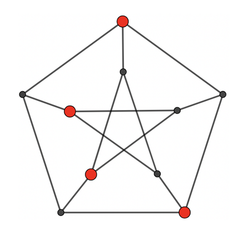


Figure 4: The Petersen graph is acritical

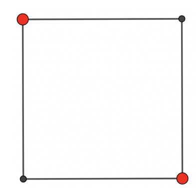


Figure 5: C_4 is acritical

Every odd cycle is critical while every even cycle is acritical.

- It suffices to prove Conjectures 1 and 2 for critical graphs.
- For any acritical graph with $\alpha\leq 8$ we have $\vartheta^{(\alpha-2)}(G)=\alpha(G)$.
- The problem of deciding whether $\vartheta^{(0)}(G)=\alpha(G)$ can be reduced in polynomial time to the same problem for acritical graphs (for fixed $\alpha(G)$).
- We can characterize the set of critical graphs with ϑ-rank $=0$:

Theorem 1. *Let G be a critical graph. Then ϑ-rank$(G)=0$ (i.e, $\vartheta^{(0)}(G)=\alpha(G)$) if and only if G is the disjoint union of cliques.*

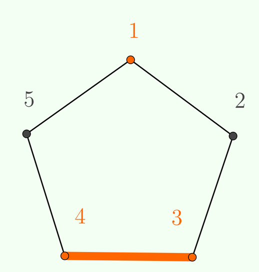
Minimizers of (M-S)

Critical edges also play a crucial role in the analysis of the minimizers of (M-S)

Theorem 2. *Let x be feasible for (M-S) with support $S:=\{i : x_i>0\}$, and $C_1,C_2,\dots,C_k$ the connected components of the graph $G[S]$. Then x is an optimal solution of (M-S) if and only if the following holds:*

- $k=\alpha(G)$,
- *C_i is a clique of critical edges of G for all $i\in[k]$,*
- $\sum_{j\in C_i}x_j=\frac{1}{\alpha(G)}$ *for all $i\in[k]$.*

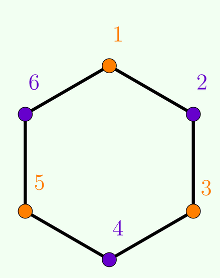
Example 3



Every optimal solution of problem (M-S) associated to C_5 has the following form (up to symmetry)

$$x_1=\frac{1}{2},\ x_3+x_4=\frac{1}{2} \text{ and } x_2=x_5=0.$$

Example 4



The only two optimal solutions of problem (M-S) associated to C_6 are

$$x_1=x_3=x_5=\frac{1}{3},\ x_2=x_4=x_6=0 \text{ and}$$
$$x_1=x_3=x_5=0,\ x_2=x_4=x_6=\frac{1}{3}.$$

Corollary 2.1. *Problem (M-S) has finitely many optimal solutions if and only if G has no critical edges.*

- The property of having finitely many minimizers is very helpful in the convergence analysis.
- We can perturb the Motzkin Strauss formulation such that it has finitely many minimizers:

$$\frac{1}{\alpha(G)}=\min\{x^T(A_c+A_G+I)x : x\in\Delta_n\}, \qquad \text{(M-S-perturbed)}$$

where A_c is the adjacency matrix by just considering the critical edges.

Theorem 3. *If there is a polynomial-time algorithm for deciding whether a standard quadratic program has finitely global minimizers, then P=NP.*

Main Result

If G is acritical then we can prove Conjecture 2.

Theorem 4. *Let G be an acritical graph, then there exists $r\in\mathbb{N}$ such that $\vartheta^{(r)}(G)=\alpha(G)$.*

Sketch of the Proof

We consider the Lasserre sum of squares hierarchy applied to problem (M-S). Let $f_G(x)=x^T(A_G+I)x$ and

$$f_G^{(r)}=\sup\lambda \text{ s.t } f_G-\lambda=\sigma_0+\sum_{i=1}^n x_i\sigma_i+\Big(\sum_{i=1}^n x_i-1\Big)q(x),$$

where σ_0,σ_i are sum of squares, $\deg(\sigma_0)\leq 2r$, $\deg(\sigma_i)\leq 2r-1$.

Then $f_G^{(r)}\leq f_G^{(r+1)}\leq\frac{1}{\alpha(G)}$ and $f_G^{(r)}\to\frac{1}{\alpha(G)}$ as $r\to\infty$.
We can link the bounds $\vartheta^{(r)}(G)$ and $f_G^{(r)}$:

For any integer $r\geq 0$ we have

$$\alpha(G)\leq\vartheta^{(2r)}(G)\leq\frac{1}{f_G^{(r)}}.$$

1) Proving finite convergence of the bounds $f_G^{(r)}$ implies finite convergence for the bounds $\vartheta^{(r)}$.
2) The classical sufficient optimality condition for nonlinear programming are satisfied at every global minimizer of (M-S) when G is acritical.
3) Using a real algebraic result of Marshall and Nie we conclude finite convergence of both hierarchies for the class of acritical graphs.

Comments and Open Questions

- The fact of having finitely many minimizers is necessary for satisfying the optimality conditions in 2).
- We can consider the hierarchy $\tilde{\vartheta}^{(r)}(G)$ derived by starting with the formulation (M-S-perturbed) instead of (M-S). The difference is that now we always have finitely many minimizers.

Theorem 5. *For any graph G there exists $r\in\mathbb{N}$ such that $\tilde{\vartheta}^{(r)}(G)=\alpha(G)$.*

Question 1. *Is it true that $\tilde{\vartheta}^{(r)}(G)=\vartheta^{(r)}(G)$ for all $r\in\mathbb{N}$?*
So far we know that it is true for $r=0$.

References

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[2] E. de Klerk and D. Pasechnik. Approximation of the stability number of a graph via copositive programming. *SIAM Journal on Optimization*, 12:875–892, 2002.

[3] N. Gvozdenović and M. Laurent. Semidefinite bounds for the stability number of a graph via sums of squares of polynomials. *Mathematical Programming*, 110:145–173, 2007.

[4] M. Laurent and L.F. Vargas. Finite convergence of sum-of-squares hierarchies for the stability number of a graph. https://arxiv.org/abs/2103.01574.