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Math 3012 – Applied Combinatorics Lecture 20

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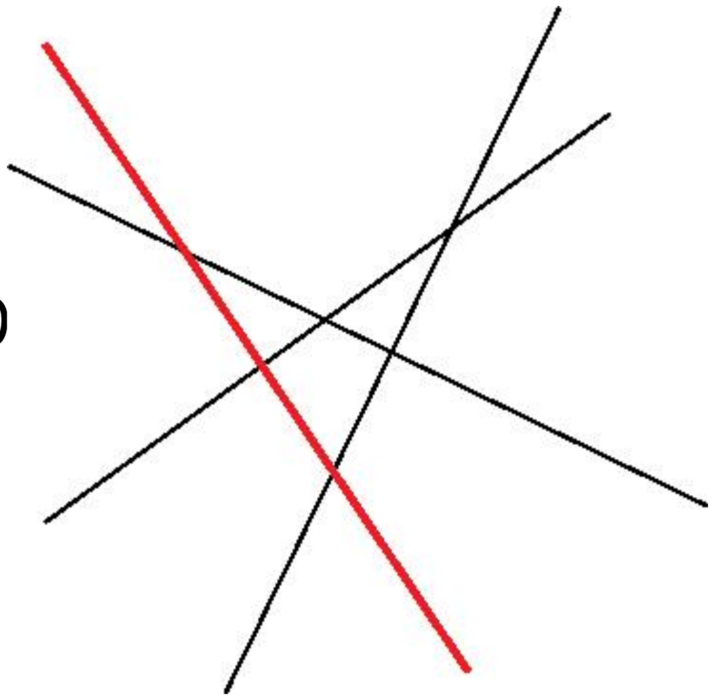
Reminder

Test 3 Tuesday, November 24, 2015. Details on material for which you will be responsible were sent by email after class the preceding Thursday. Again, I ask all of you to study hard. Experience shows that the closing portion of this course has most content. The concepts and techniques will have lasting value.

Review of Recurrence Equations (1)

Problem Let $r(n)$ denote the number of regions determined by n lines that intersect in general position.



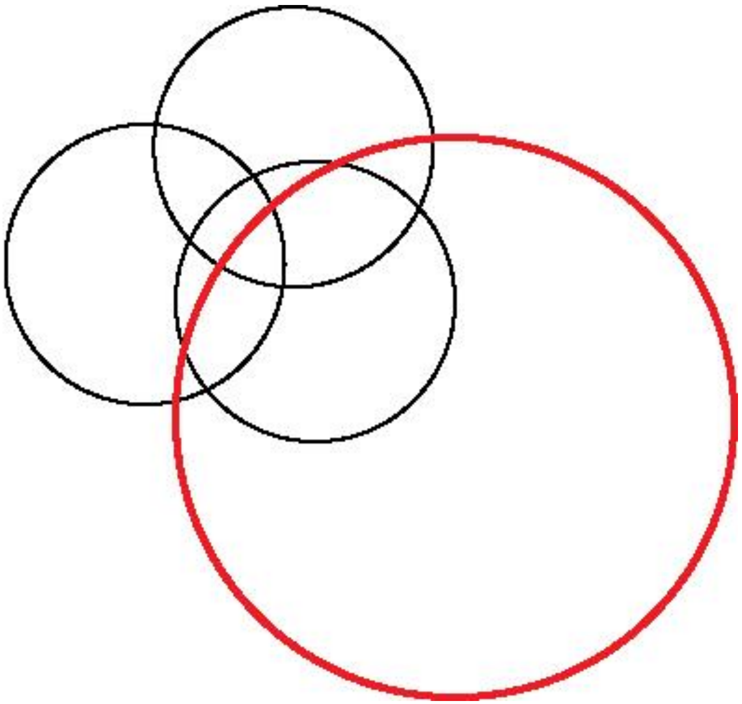
Solution

$r(1) = 2$

$r(n + 1) = r(n) + n+1$ when $n \geq 0$

Review of Recurrence Equations (2)

Problem Let s(n) denote the number of regions determined by n circles that intersect in general position.



Solution

$s(1) = 2$

$s(n+1) = s(n) + 2n$ when $n \geq 0$.

Review of Recurrence Equations (3)

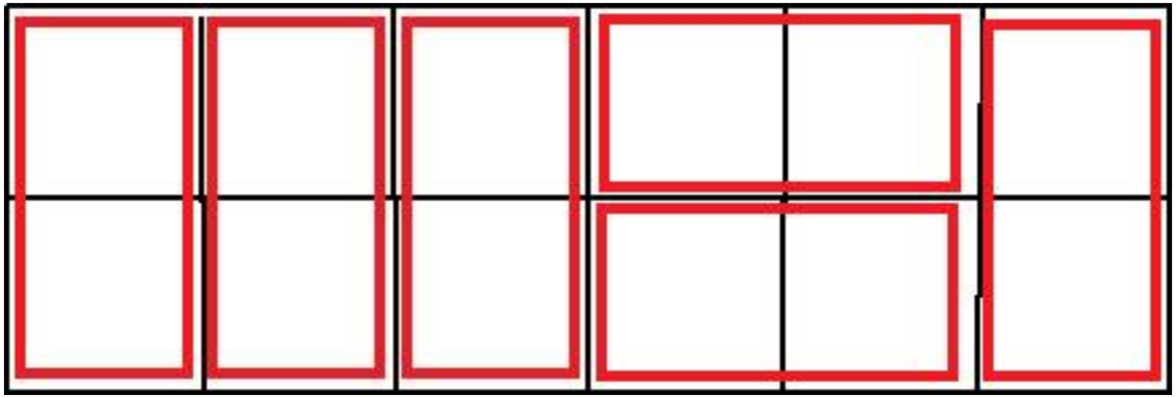
Problem Let $t(n)$ denote the number of ways to tile a $2 \times n$ grid with dominoes of size 1×2 and 2×1.

Solution

$t(1) = 1$

$t(2) = 2$

$t(n+2) = t(n+1) + t(n)$ when $n \geq 0$.



Review of Recurrence Equations (4)

Problem Let $u(n)$ denote the number of ternary sequences that do not contain 01 in consecutive positions.

Solution

$u(1) = 3$

$u(2) = 8$

$u(n+2) = 3u(n+1) - u(n)$

Review of Recurrence Equations (5)

Summary The recurrence equations in the last four examples are:

$$r(n+1) - r(n) = n+1$$

$$s(n+1) - s(n) = 2n$$

$$t(n+2) - t(n+1) - t(n) = 0$$

$$u(n+2) - 3u(n+1) + u(n) = 0$$

Developing a General Framework (1)

Observation We consider the family **V** of all functions which map the set **Z** of all integers (positive, negative and zero) to the set **C** of complex numbers. This is a more general framework than we first studied, but as will become clear, we need this additional structure to make the form of general solutions relatively easy to obtain.

Note Each of the four examples presented above have involved functions with range and domain being the set **N** of positive integers, so **V** is a more general setup.

Developing a General Framework (2)

Fact The family **V** is an infinite dimensional vector space over the field ***C*** of complex numbers, with $(f + g)(n) = f(n) + g(n)$ and $(a\,f)(n) = a(f(n))$.

Note Students should spot the "operator overloading" in these two equations, even when one of the two operators (multiplication) is indicated simply by adjacent symbols, one a scalar and the other a vector.

Note The "zero" of **V** is the constant function which maps all integers to the "zero" in ***C***.

Developing a General Framework (3)

Observation We will first focus on homogeneous linear recurrence equations. These have the following form:

$$a_0 f(n+d) + a_1 f(n+d-1) + a_2 f(n+d-2) + \dots + a_{d-1} f(n+1) + a_d f(n) = 0$$

Note The coefficients $a_0, a_1, a_2, \dots, a_d$ are complex numbers. Without loss of generality $a_0 \neq 0$. For the time being, we will also assume that $a_d \neq 0$.

Developing a General Framework (4)

Example A homogeneous equation:

$$(2+3i)g(n+3) - (8-7i)g(n+2) + 42g(n+1) - (5i)g(n) = 0$$

Example A non-homogeneous equation:

$$(2+3i)g(n+3) - (8-7i)g(n+2) + 42g(n+1) - (5i)g(n) = (2-i)(3+i)^n + 12n^3$$

Remark In order to fully understand the homogeneous case, we will need to discuss the non-homogeneous case concurrently.

Developing a General Framework (5)

Alternate Notation We define the advancement operator A on the vector space V by the rule $A\, f(n) = f(n+1)$. Note that $A^2 f(n) = f(n+2)$, $A^3 f(n) = f(n+3)$, etc. So our linear homogeneous equation

$$a_0 f(n+d) + a_1 f(n+d-1) + a_2 f(n+d-2) + \ldots + a_{d-1} f(n+1) + a_d f(n) = 0$$

can then be rewritten as:

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \ldots + a_{d-1} A + a_d)\, f(n) = 0.$$

Remark The "polynomial form" is significant!

Developing a General Framework (6)

Theorem The set S of all solutions to a homogeneous linear recurrence equation of the form:

$$(a_0A^d + a_1A^{d-1} + a_2A^{d-2} + \ldots + a_{d-1}A + a_d)\, f_n = 0$$

is a d-dimensional subspace of $\mathbf{V}$ provided both a_0 and a_d are non-zero

Conclusion The solution space can be specified entirely just by providing a basis for the subspace S.

The Case $d = 1$

Theorem Let a_0 and a_1 be non-zero complex numbers, and set $r = (-a_1/a_0)$. Then the solution space S of the advancement operator equation $(a_0A + a_1)f(n) = 0$ is a 1-dimensional subspace of V and the function r^n is a basis, i.e., every solution is of the form $f(n) = c_1 r^n$ where c_1 is a constant.

Proof Let f be any solution to $(a_0A + a_1)f(n) = 0$, and let $c_1 = f(0)$. We show that $f(n) = c_1 r^n$ for all integers n. We first show that $f(n) = c_1 r^n$ when $n \geq 0$. We do this by induction on n.

The Case d = 1 (Part 2)

Base Case The base case is $n = 0$, where the left hand side is $f(0) = c_1$, and the right hand side is $c_1 r^0$. But since $r \neq 0$, the right hand side is $c_1 \cdot 1 = c_1$. So the statement holds when $n = 0$.

The Inductive Step Now assume that $f(k) = c_1 r^k$ for some $k \geq 0$. Since $(a_0 A + a_1)f(n) = 0$ for all integers n, we know that:

$$(a_0 A + a_1)f(k) = 0$$
$$(a_0 f(k+1) + a_1 f(k) = 0$$
$$f(k+1) = (-a_1/a_0)c_1 r^k$$
$$f(k+1) = r\ c_1\ r^k$$
$$f(k+1) = c_1 r^{k+1}$$

The Case d = 1 (Part 3)

Negative Integers It remains only to show that $f(n) = c_1 r^n$ for all integers $n \leq 0$. This is equivalent to showing that $f(-n) = c_1 r^{-n}$ for all $n \geq 0$. This is done by induction and the argument is a trivial modification of what we have just done.

Conclusion We have verified the assertion that the solution space to the homogeneous equation

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \ldots + a_{d-1} A + a_d)\, f(n) = 0$$

with a_0 and a_d non-zero is a d-dimensional subspace of **V** when $d = 1$.

Towards the General Case (1)

Exercise Note that

$A^2 + 2A - 35 = (A + 7)(A - 5)$

Example The functions $(-7)^n$ and 5^n are solutions to the equation:
$(A^2 + 2A - 35) f(n) = 0.$

Observation If $r \neq 0$ and r is a root of the advancement operator polynomial, then r^n is a solution.

Towards the General Case (2)

Exercise Show that

$A^2 + (-12 + i)A + 41 - i = (A - 5 - 2i)(A - 7 + 3i)$

Example The functions $(5 + 2i)^n$ and $(7 - 3i)^n$ are solutions to the equation:
$(A^2 + (-12 + i)A + 41 - i) f(n) = 0.$

Observation If $r \neq 0$ and r is a root of the advancement operator polynomial, then r^n is a solution.

Towards the General Case (3)

Example Note that $A^2 - 10A + 25 = (A - 5)^2$.

Also note that the functions 5^n and $n5^n$ are solutions to the equation:

$$(A - 5)^2 f(n) = 0.$$

Observation If $r \neq 0$ and r is a root of multiplicity 2, then r^n and $n\, r^n$ are solutions.

Towards the General Case (4)

Example The functions $(5 - 2i)^n$ and $n(5 - 2i)^n$ are solutions to the equation:

$$(A - 5 + 2i)^2 f(n) = 0.$$

Observation If $r \neq 0$ and r is a root of multiplicity 2, then r^n and $n r^n$ are solutions.

Towards the General Case (5)

Lemma If $p(A)$ is a polynomial in the advancement operator A, $r \neq 0$ and r is a root of multiplicity m, then each of the following functions is a solution of the equation: $p(A)\, f(n) = 0$

$$r^n \quad n\, r^n \quad n^2\, r^n \quad n^3\, r^n \quad n^4\, r^n \quad \dots \quad n^{m-1}\, r^n$$

Proof We will outline the proof in Thursday's lecture.

Towards the General Case (6)

Example The general solution to

$$((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2)\ f(n) = 0$$

is:

$$f(n) = c_1 3^n + c_2 n 3^n + c_3 n^2 3^n + c_4 n^3 3^n$$
$$+ c_5(7 - 2i)^n + c_6 n(7 - 2i)^n + c_7 n^2(7 - 2i)^n$$
$$+ c_8(-5 + 8i)^n + c_9 n(-5 + 8i)^n$$

Towards the General Case (7)

Example The solution space to:

$$((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2)\ f(n) = 0$$

is a 9-dimensional subspace of V and the following functions are a basis:

3^n $n3^n$ $n^2 3^n$ $n^3 3^n$

$(7 - 2i)^n$ $n(7 - 2i)^n$ $n^2(7 - 2i)^n$

$(-5 + 8i)^n$ $n(-5 + 8i)^n$

Analogies with Partial Fractions

Example Given a proper rational function $p(x)/q(x)$ whose denominator polynomial $q(x)$ can be factored as

$$q(x) = (x - 3)^3 x^2(x^2 + 2x + 9)$$

there are constants $c_1, c_2, c_3, c_4, c_5, c_6$ and c_7 so that

$$\begin{aligned} p(x)/q(x) = {} & c_1/(x - 3) + c_2/(x - 3)^2 + c_3/(x - 3)^3 \\ & + c_4/x + c_5/x^2 \\ & + c_6/(x^2 + 2x + 9) + c_7x/(x^2 + 2x + 9) \end{aligned}$$

Analogies with Differential Equations

Example Let D be the differential operator, i.e., D f is the derivative of f. Then the solution to the equation:

$$(D - 3)^3 D^2(D^2 + 16) f = 0$$

has the form:

$$f(x) = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x}$$
$$+ c_4 + c_5 x$$
$$+ c_6 e^{4i} + c_7 e^{-4i}$$

where c_6 and c_7 are complex conjugates.

The Non-Homogeneous Case

Theorem Let $p(A)\, f = g$ be a non-homogeneous equation. If h_0 is any solution to this equation, then the general solution is $h_0 + f$ where f is a solution to the associated homogeneous equation $p(A)\, f = 0$.

Note The proof of this theorem is relatively straightforward.

Terminology The function h_0 is referred to as a particular solution to $p(A)\, f = g$.

The Non-Homogeneous Case (2)

Example For the non-homogeneous equation $(A - 3)\, f(n) = 8\,(5)^n$, the function $h_0 = 4 \cdot 5^n$ is a particular solution. Accordingly, the general solution has the form:

$$f(n) = c_1 3^n + 4 \cdot 5^n$$