Viscous effects in a collisional tokamak plasma with strong rotation

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The full viscosity tensor for an axisymmetric toroidal plasma in the collisional regime (with strong rotation) is calculated, including gyroviscosity and $O(\epsilon)$ poloidal variations over the flux surface. It is shown that the resulting viscous force is of sufficient magnitude to account for the radial transfer of toroidal momentum that must be inferred in order to explain the rotation measurements in tokamak experiments. The consequences of a viscous force of this form and magnitude on particle transport and on the evolution of toroidal and poloidal rotation velocities are discussed.

I. INTRODUCTION

Measured[1–4] toroidal rotation velocities in tokamak experiments with unbalanced neutral beam injection indicate that the radial transport of toroidal momentum apparently occurs at a rate one to two orders of magnitude faster than can be accounted for by standard (i.e., proportional to the self-collision frequency) neoclassical perpendicular viscosity.[5–8] This has led to the feeling that an anomalous mechanism was responsible for the "enhanced" radial transport of toroidal momentum and correspondingly "reduced" momentum confinement times in these experiments.

We have been motivated by this situation to derive the full viscosity tensor with full toroidal geometry and all pieces of Braginskii's stress tensor, including the gyroviscous tensor elements and taking into account the $O(\epsilon)$ poloidal density variations[9] produced by strong rotation ($V_\phi \simeq V_{\text{th}}$). The formalism is developed for a collisional plasma, but some of the consequences are of more general applicability. We confirm that the perpendicular viscous forces, which are proportional to the self-collision frequency, are too small to account for the rotation measurements, but we find that the gyroviscous forces, as modified by the toroidal geometry effects, are of the proper magnitude to explain the momentum confinement time inferred from these experiments. The effects of this large gyroviscous force in coupling the time evolution of toroidal and poloidal rotation velocities and in driving a radial particle flux are also examined.

II. RATE-OF-STRAIN TENSOR

The elements of the general rate-of-strain tensor of fluid theory may be written

$$W_{\alpha\beta} \equiv \hat{n}_\alpha \cdot \nabla \mathbf{V} \cdot \hat{n}_\beta + \hat{n}_\beta \cdot \nabla \mathbf{V} \cdot \hat{n}_\alpha - \tfrac{2}{3}\delta_{\alpha\beta}\nabla \cdot \mathbf{V}, \tag{1}$$

where $\mathbf{V}$ is the fluid velocity and $\hat{n}_\alpha$ is the unit vector in the α direction. The elements of the rate of strain tensor can be written generally as[10]

$$W_{\alpha\beta} = \left(\frac{\partial V_\beta}{\partial l_\alpha} + \sum_k \Gamma^{\alpha}_{\beta k} V_k\right) + \left(\frac{\partial V_\alpha}{\partial l_\beta} + \sum_k \Gamma^{\beta}_{\alpha k} V_k\right) - \frac{2}{3}\delta_{\alpha\beta}\nabla \cdot \mathbf{V}, \tag{2}$$

where dl_α are the differential elements in the α-coordinate directions and $\Gamma^{\beta}_{\alpha k}$ are the Christoffel symbols (see the Appendix). We adopt an orthogonal (ψ,p,ϕ) flux surface coordinate system with $dl_\psi = h_\psi\, d_\psi = (RB_p)^{-1}\, d\psi$, $dl_p = h_p\, d_p$, $dl_\phi = h_\phi\, d_\phi = R\, d\phi$, where R is the major radius and B_p is the poloidal magnetic field. We further make use of toroidal symmetry. The elements of the traceless rate-of-strain tensor in this coordinate system are

$$\begin{aligned}
W_{\psi\psi} &= -\frac{2}{3}\frac{\partial V_p}{\partial l_p} + \left(\frac{4}{3}RB_p\frac{\partial (RB_p)^{-1}}{\partial l_p} - \frac{2}{3}\frac{1}{R}\frac{\partial R}{\partial l_p}\right)V_p,\\
W_{\psi p} &= h_p\frac{\partial}{\partial l_\psi}(V_p h_p^{-1}) = W_{p\psi},\\
W_{\psi\phi} &= R\frac{\partial}{\partial l_\psi}(V_\phi R^{-1}) = W_{\phi\psi},\\
W_{pp} &= \frac{4}{3}\frac{\partial V_p}{\partial l_p} - \frac{2}{3}\left(RB_p\frac{\partial (RB_p)^{-1}}{\partial l_p} + \frac{1}{R}\frac{\partial R}{\partial l_p}\right)V_p,\\
W_{p\phi} &= \frac{R\partial (V_\phi R^{-1})}{\partial l_p} = W_{\phi p},\\
W_{\phi\phi} &= -\frac{2}{3}\frac{\partial V_p}{\partial l_p} + \left(\frac{4}{3}\frac{1}{R}\frac{\partial R}{\partial l_p} - \frac{2}{3}RB_p\frac{\partial (RB_p)^{-1}}{\partial l_p}\right)V_p.
\end{aligned} \tag{3}$$

III. VISCOUS STRESS TENSOR

We follow Braginskii[11] in computing the elements of the viscous stress tensor for a plasma in a magnetic field from the corresponding rate-of-strain tensor elements:

$$\begin{aligned}
\Pi_{\alpha\beta} &= -\eta_0 W^0_{\alpha\beta} - [(\eta_1 W^1_{\alpha\beta} + \eta_2 W^2_{\alpha\beta})] + [(\eta_3 W^3_{\alpha\beta} + \eta_4 W^4_{\alpha\beta})]\\
&\equiv \Pi^0_{\alpha\beta} + \Pi^{12}_{\alpha\beta} + \Pi^{34}_{\alpha\beta},
\end{aligned} \tag{4}$$

where the prescription for construction of the $W^n_{\psi\beta}$ ($n = 0,1,2,3,4$) from the rate-of-strain tensor elements of Eq. (3) is given in the Appendix and the viscosity coefficients are given by Braginskii.[11] The stress tensor elements in this form naturally decompose into a "parallel" term $\Pi^0_{\alpha\beta}$, a "perpendicular" term $\Pi^{12}_{\alpha\beta}$, and a "gyroviscous" term $\Pi^{34}_{\alpha\beta}$. The

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particle motions producing these stresses are discussed in Ref. 12. The parallel viscosity coefficient η_0 scales inversely with self-collision frequency; the perpendicular viscosity coefficients η_1 and η_2 scale directly with collision frequency and inversely as the square of the magnetic field; and the gyroviscous coefficients, η_3 and η_4, are independent of collision frequency and scale inversely with the magnetic field.

We assume $f_\phi \equiv |B_\phi|/|B| \simeq 1$, $f_p \equiv |B_p|/|B| \ll 1$. The viscosity tensor elements can then be represented as given in Table I.

We note that Mikhailovski and Tsypin[13] have also recently generalized the Braginskii representative of $\mathbf{\Pi}$ to general curvilinear coordinates. In this respect, Table I can be considered an explicit evaluation of their general result for tokamak flux surface coordinates, although we worked directly from the Braginskii representation.

IV. TOROIDAL VISCOUS FORCE

The flux-surface-averaged (denoted by $\langle\ \ \rangle$) toroidal momentum balance on a tokamak plasma, summed over species, can be written

$$\sum_j \left\langle R^2 \nabla\phi \cdot n_j m_j \frac{d\mathbf{V}_j}{dt} \right\rangle = \sum_j \langle R^2 \nabla\phi \cdot \mathbf{M}_j \rangle - \sum_j \langle R^2 \nabla\phi \cdot \nabla \cdot \mathbf{\Pi}_j \rangle, \tag{5}$$

where n and m are the number density and mass of the plasma ions, M is the momentum density input to the plasma by the unbalanced beam injection, and $d\mathbf{V}/dt$ is the convective derivative.

The toroidal component of the viscous force can be written, by specializing the representation in general curvilinear coordinates[10] to flux-surface coordinates, as

$$R^2 \nabla\phi \cdot \nabla \cdot \mathbf{\Pi} = \frac{1}{Rh_p} \frac{\partial}{\partial l_\psi} (R^2 h_p \Pi_{\psi\phi}) + B_p \frac{\partial}{\partial l_p} \left(\frac{R \Pi_{p\phi}}{B_p} \right). \tag{6}$$

Recall that $\Pi_{\alpha\beta} \equiv \hat{n}_\alpha \cdot \mathbf{\Pi} \cdot \hat{n}_\beta$.

The flux-surface average of this force is

$$\langle R^2 \nabla\phi \cdot \nabla \cdot \mathbf{\Pi} \rangle = \left\langle \frac{1}{Rh_p} \frac{\partial}{\partial l_\psi} (R^2 h_p \Pi_{\psi\phi}) \right\rangle. \tag{7}$$

We note that, since $\mathbf{\Pi}$ is a symmetric tensor,

$$\langle R^2 \nabla\phi \cdot \nabla \cdot \mathbf{\Pi} \rangle = \frac{1}{V'} \frac{\partial}{\partial\psi} (V' \langle R^2 \nabla\phi \cdot \mathbf{\Pi} \cdot \nabla\psi \rangle).$$

Making use of Table I, we find that the parallel (η_0) viscosity contribution to the toroidal viscous force vanishes, that the perpendicular $(\eta_1\eta_2)$ viscosity contribution can be written (to leading terms in $f_p \ll 1$) as

$$\langle (R^2 \nabla\phi \cdot \nabla \cdot \mathbf{\Pi})_{12} \rangle \simeq - \left\langle \frac{1}{Rh_p} \frac{\partial}{\partial l_\psi} \left(R^3 h_p \eta_2 \frac{\partial (V\phi R^{-1})}{\partial l_\psi} \right) \right\rangle, \tag{8}$$

and the gyroviscous $(\eta_3\eta_4)$ contribution can similarly be written as

$$\langle (R^2 \nabla\phi \cdot \nabla \cdot \mathbf{\Pi})_{34} \rangle \simeq - \left\langle \frac{1}{Rh_p} \frac{\partial}{\partial l_\psi} \left(R^3 h_p \eta_4 \frac{\partial (V\phi R^{-1})}{\partial l_p} \right) \right\rangle. \tag{9}$$

We note that if the plasma rotated as a rigid body, then $V_\phi(\psi,p) = R(\psi,p)\,\Omega$, and both of the viscous forces would vanish. Thus, it is the departure from rigid-body rotation within a flux surface $\partial(V_\phi R^{-1})/\partial l_p \neq 0$ that drives the gyroviscous force, and it is the departure from radially uniform rigid-body rotation, $\partial(V_\phi R^{-1})/\partial l_\psi \neq 0$, that drives the perpendicular viscous force.

TABLE I. The viscosity tensor elements, where $A_0 = 2\{ -\frac{1}{3}(\partial V_p/\partial l_p) + [(1/R)(\partial R/\partial l_p) + \frac{1}{3}(1/B_p)(\partial B_p)/\partial l_p] V_p + f_p R [\partial(V_\phi R^{-1})/\partial l_p] \}$.

	Parallel $\Pi^0_{\alpha\beta} \equiv -\eta_0 W^0_{\alpha\beta}$	Perpendicular $\Pi^{12}_{\alpha\beta} \equiv -(\eta_1 W^1_{\alpha\beta} + \eta_2 W^2_{\alpha\beta})$	Gyroviscous $\Pi^{34}_{\alpha\beta} \equiv (\eta_3 W^3_{\alpha\beta} + \eta_4 W^4_{\alpha\beta})$
$\Pi_{\psi\psi}$	$\frac{1}{2}\eta_0 A_0$	$\eta_1\left((RB_p)^{-1}\frac{\partial(RBV_p)}{\partial l_p} - f_p R \frac{\partial(V_\phi R^{-1})}{\partial l_p}\right)$	$-\eta_3\left(h_p \frac{\partial(V_p h_p^{-1})}{\partial l_\psi} - f_p R \frac{(V_\phi R^{-1})}{\partial l_\psi}\right)$
$\Pi_{\psi p} = \Pi_{p\psi}$	0	$-\eta_1 h_p \frac{\partial(V_p h_p^{-1})}{\partial l_\psi}$	$-\eta_3 (RB_p)^{-1} \frac{\partial(RB_p V_p)}{\partial l_p}$
		$+(\eta_2 - \eta_1) f_p R \frac{\partial(V_\phi R^{-1})}{\partial l_\psi}$	$-(\eta_4 - \eta_3) f_p R \frac{\partial(V_\phi R^{-1})}{\partial l_p}$
$\Pi_{\psi\phi} = \Pi_{\phi\psi}$	0	$-\eta_2 R \frac{\partial(V_\phi R^{-1})}{\eta l_\psi}$	$-\eta_4 R \frac{\partial(V_\phi R^{-1})}{\partial l_p}$
Π_{pp}	$\frac{1}{2}\eta_0 A^0$	$-\eta_1 (RB_p)^{-1} \frac{\partial(RB_p V_p)}{\partial l_p}$	$\eta_3 h_p \frac{\partial(V_p h_p^{-1})}{\partial l_\psi}$
		$+(\eta_1 - 2\eta_2) f_p R \frac{\partial(V_\phi R^{-1})}{\partial l_p}$	$+(2\eta_4 - \eta_3) f_p R \frac{\partial(V_\phi R^{-1})}{\partial l_\psi}$
$\Pi_{p\phi} = \Pi_{\phi p}$	$-\frac{3}{2}\eta_0 f_p A^0$	$-\eta_2 R \frac{\partial(V_\phi R^{-1})}{\partial l_p}$	$\eta_4 R \frac{\partial(V_\phi R^{-1})}{\partial l_\psi}$
$\Pi_{\phi\phi}$	$-\eta_0 A^0$	$2\eta_2 f_p R \frac{\partial(V_\phi R^{-1})}{\partial l_p}$	$-2\eta_4 f_p R \frac{\partial(V_\phi R^{-1})}{\partial l_\psi}$

To leading order, V_ϕ is uniform over the flux surface. Hogan[8] has recently shown that the viscosity itself drives a higher-order $[O(\nu)]$, poloidally asymmetric flow that, when used to evaluate Eq. (9), contributes a term that is $2.31q^2$ times Eq. (8), thus constituting an effective "Pfirsch–Schlüter" type factor. We[9] have shown that with strong rotation ($V_\phi \sim V_{\rm th}$), inertial effects drive an $O(\epsilon)$ poloidal variation in the density that in turn produces a higher-order $[O(\epsilon)]$, poloidally asymmetric flow. We shall now evaluate Eq. (9) in the presence of strong rotation and a poloidally asymmetric flow of order ϵ.

In the large aspect ratio, low-β, circular flux-surface approximation ($\epsilon \equiv r/R_0 \ll 1$), $p \to \theta$, $h_p \to 1/r$, $\psi \to r$, and $R = R_0 \times (1 + \epsilon \cos\theta)$. A strong toroidal rotation produces poloidal variations[9]:

$$V_\phi = V_\phi^0(1 + \widetilde{V}_c \cos\theta + \widetilde{V}_s \sin\theta),$$

$$\eta = \eta^0(1 + \tilde{n}_c \cos\theta + \tilde{n}_s \sin\theta).$$

The $\tilde{x}$ quantities are of the order $O(\epsilon)$. In this approximation, Eqs. (8) and (9) reduce to

$$\langle R^2 \nabla\phi \cdot \nabla \cdot \Pi_{12} \rangle \simeq -R_0 \frac{1}{r}\frac{\partial}{\partial r}\left(\eta_2^0 r \frac{\partial V_\phi^0}{\partial r}\right) + O(\epsilon^2),$$

$$\langle R^2 \nabla\phi \cdot \nabla \cdot \Pi_{34} \rangle \simeq -\frac{1}{2}\epsilon^2 R_0 \left(\frac{1}{r}\frac{\partial}{\partial r}(\eta_4^0 V_\phi^0)\right. \times\left\{\left[3 + \left(\frac{\tilde{n}_c}{\epsilon}\right)\right]\left(\frac{\widetilde{V}_s}{\epsilon}\right) + \left[1 - \left(\frac{\widetilde{V}_c}{\epsilon}\right)\right]\left(\frac{\tilde{n}_s}{\epsilon}\right)\right\}\Bigg) \tag{10}$$

$$\equiv -\tfrac{1}{2} R_0^{-1} \eta_4^0 V_\phi^0 [G(r)\widetilde{\Theta}] + O(\epsilon^3), \tag{11}$$

where

$$G(r) \equiv -\frac{r}{\eta_4^0 V_\phi^0}\frac{\partial(\eta_4^0 V_\phi^0)}{\partial r},$$

$$\widetilde{\Theta} \equiv \left\{\left[3 + \left(\frac{\tilde{n}_c}{\epsilon}\right)\right]\left(\frac{\widetilde{V}_s}{\epsilon}\right) + \left[1 - \left(\frac{\widetilde{V}_c}{\epsilon}\right)\right]\left(\frac{\tilde{n}_s}{\epsilon}\right)\right\}. \tag{12}$$

The quantity $\widetilde{\Theta}$ is order unity when there are $O(\epsilon)$ poloidal variations in toroidal velocity and density.

V. INTERPRETATION OF ROTATION MEASUREMENTS

If we associate the viscous toroidal force with the momentum loss term used to interpret the rotation experiments,[1–4] nmV_ϕ/τ_ϕ (where τ_ϕ is the momentum confinement time), or with the equivalent "drag" force that we previously introduced[9,14] to model radial transfer of toroidal momentum, $nm\nu_d V_\phi$ (where ν_d is a radial momentum transfer or "drag" frequency), and make use of Eqs. (10) and (11), we obtain

$$n_0 m \nu_d V_\phi^0 \equiv \frac{nmV_\phi^0}{\tau_\phi} = -\frac{1}{r}\frac{\partial}{\partial r}\left(\eta_2^0 r \frac{\partial V_\phi^0}{\partial r}\right) - \frac{1}{2}\epsilon^2 \frac{1}{r}\frac{\partial}{\partial r}(\eta_4^0 V_\phi^0)\widetilde{\Theta}. \tag{13}$$

Using Braginskii's[11] viscosity coefficients

$$\eta_4^0 \simeq n^0 Tm/ZeB, \quad \eta_2^0 \simeq \eta_4^0/\Omega\tau, \tag{14}$$

where $\Omega = (m/ZeB)$ is the gyrofrequency and τ is the ion self-collision frequency, we find that for the conditions in present tokamak plasmas $\Omega\tau \simeq 10^{-3}$–$10^4$, so that the ratio of the gyroviscous force to the perpendicular viscous force is $\simeq \epsilon^2\widetilde{\Theta} \times (10^3$–$10^4)$. Thus, for $O(\epsilon)$ poloidal variations over the flux surface, the gyroviscous force is $\sim 10^2$ larger than the perpendicular viscous force, allowing us to ignore the first term on the rhs of Eq. (13) in solving for

$$\tau_\phi^{-1} \equiv \nu_d = -\frac{\epsilon^2\widetilde{\Theta}}{2ZeB}\left[\frac{1}{n^0 V_\phi^0}\left(\frac{1\partial}{r\partial r} n^0 T V_\phi^0\right)\right]. \tag{15a}$$

We thus conclude that the dominant radial transport of toroidal momentum is caused by the gyroviscous mechanism and scales like η_4, which is independent of collision frequency. Thus, Eq. (15) should be applicable to all collision regimes. We note that this result differs from earlier conclusions that the radial transport of toroidal momentum varied inversely[7,15] or directly[8] with the collision frequency. Tsang and Frieman[7] apparently calculated a parallel viscosity term ($\sim \eta_0$) which, as we will see, enters into the parallel viscous force but, as we have shown, does not enter into the toroidal viscous force. Grimm and Johnson[15] worked out the viscosity tensor in tokamak geometry, but only considered the parallel viscous force. Hogan[8] correctly identified both the perpendicular and gyroviscosity terms. His result is equivalent to our Eqs. (8) and (9) within the limits of his approximations. However, his approximate evaluation of the poloidal variation of V_ϕ neglected inertial forces which can produce $O(\epsilon)$ variations, and his results are based on smaller variations produced by the viscous forces, leading him to conclude that the dominant viscous force scaled with collision frequency.

Momentum confinement times inferred from toroidal rotation measurements and momentum balance considerations have been about two orders of magnitude smaller than could be accounted for by neoclassical perpendicular viscosity.[1–4] We have shown that the gyroviscous force is about two orders of magnitude greater than the perpendicular force in a strongly rotating tokamak plasma. Thus, the momentum confinement time predicted based upon gyroviscous effects must be the same order of magnitude as the momentum confinement time inferred from the rotation experiments.

Equation (15a) can be rewritten in a form that provides more physical insight:

$$\tau_\phi^{-1} \equiv \nu_d = (T\widetilde{\Theta}/2R_0^2 ZeB)G(r), \tag{15b}$$

where $G(r)$ is a geometrical factor that depends upon the radial profile of $n_0 T V_\phi^0$ [see Eqs. (12) and (14)]. For example, when the profile $\sim (1 - x^2)^n$, then $G = 2nx^2/(1 - x^2)$, where $x = r/a$ is the normalized radius, so that $|G|$ is order unity. We have previously calculated[9,16] $O(\epsilon)$ density variations over the flux surface in strongly rotating plasmas with parameters typical of ISX-B[3] and PLT,[1,2] and substantial impurity density variations have been measured in PDX.[17] Thus, the product $\widetilde{\Theta}G(r)$ is order unity in a strongly rotating plasma. Lacking detailed knowledge, we set $\widetilde{\Theta}G(r) = 1$ in the following comparisons.

The scaling $\tau_\phi \sim (R_0^2 BT_i^{-1})$ given by Eq. (15b) is consistent with the experimental observation[4] that the momentum confinement time is greater in PDX ($R_0 = 1.45$ m, $B = 2.2$ T, $T_i \simeq 800$ eV) than in PLT ($R_0 = 1.3$ m, $B = 2.5$ T, $T_i \simeq 900$ eV), as well as the observation[18] that the momentum confinement time is less in ISX-B ($R_0 = 93$ cm, $B = 1.4$ T, $T_i \simeq 500$ eV) than in PLT.

We have compared the prediction of Eq. (15b) with two experimentally determined values of the momentum confinement time in ISX-B.[18] In the first case, the steady-state toroidal rotation velocity was measured for an oxygen impurity, then a momentum confinement time was inferred by assuming that all of the plasma ions were rotating with the same velocity and balancing the momentum input from the beam with the momentum loss rate. The inferred experimental value of the momentum confinement time thus determined was 16 msec. Applying Eq. (15b) to a composite ion species with an effective charge $\overline{Z} = 2.5$ and an average temperature $\overline{T}_i = 500$ eV[16] results in a predicted momentum confinement time of 12 msec.

In the second ISX-B experiment, the decay of the Ne $^{\rm IX}$ toroidal rotation velocity when the beam was shut off led to a preliminary experimental confinement time of 35 msec for the center of the plasma.[18] Applying Eq. (15b) to an ion with charge $Z = 9$ and a central ion temperature of $T_i(0) = 800$ eV[18] results in a predicted momentum confinement time of 28 msec.

In PLT, momentum confinement times of 10–30 msec were inferred[1,2] from measured toroidal rotation velocities and balancing momentum input with momentum losses on the bulk plasma. Applying Eq. (15b) to a composite ion species with an effective charge $Z = 2.5$ and an average ion temperature $T_i = 900$ eV[19] results in a predicted momentum confinement time of 23 msec.

The decay of the toroidal rotation velocity of centrally peaked Ti $^{\rm XXI}$ after the beams were turned off in PDX was used to determine[4] a central momentum confinement time of 80–100 msec for PDX. These experiments were performed for 3.5 MW of beam power. For similar plasma parameters and 7.2 MW of beam power, the central ion temperature in PDX was 6 keV.[20] Applying Eq. (15b) to a titanium ion with charge $Z = 21$ and temperature $T_i(0) = 2.5$ keV results in a predicted momentum confinement time of 78 msec.

Thus, there is good agreement between the experimentally determined confinement time and the prediction of Eq. (15b) over a range of major radii (93–145 cm), magnetic field (1.4–2.5 T), ion charge (2.5–21), and ion temperature (500–2500 eV). These results are summarized in Table II.

In PLT, the toroidal rotation velocity was observed[1,2] to vary linearly with the beam momentum input, whereas in ISX-B the toroidal rotation velocity was saturated.[18] We have made some preliminary calculations of the poloidal variation of the impurity density in these experiments, based on the theory of Ref. 16. We predict that the θ component of the impurity density variation in *PLT* is $O(\epsilon)$ and does not vary greatly over the range of beam powers for which the rotation experiments were performed. Since $\widetilde{V} \sim \tilde{n}^{-1}$, this would lead us to conclude from Eq. (15b) that, aside from temperature effects, the momentum confinement time would be roughly constant over the range of the PLT rotation experiments. This is consistent with the observed linear dependence of rotation velocity on momentum input. On the other hand, our calculations for ISX-B indicate that $\widetilde{\Theta}$ could increase significantly with increasing momentum input over the range of the rotation experiments, which is consistent with a saturation of the toroidal rotation velocity.

Oscillations almost 180° out of phase between the toroidal rotation velocity and the temperature have been observed in ISX-B.[18] Equation (15a) predicts that an increase in temperature would lead to a decrease in momentum confinement time, which would be followed by a decrease in toroidal rotation velocity, and conversely. The experimental result is $V_1/V_2 \simeq 1.5$ and $T_2/T_2 \simeq 2$, which is not inconsistent with the prediction of Eq. (15b) that $V_1/V_2 = T_2/T_1$ when possible $G(r)$ and $\widetilde{\Theta}$ variations are neglected.

VI. PARALLEL VISCOUS FORCE

The parallel viscous force in flux surface coordinates is represented as

$$\begin{aligned}\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi} = B_p\bigg[&\frac{1}{Rh_p^2}\frac{\partial}{\partial l_\psi}(Rh_p^2\Pi_{\psi p}) + B_p\frac{\partial}{\partial l_p}\left(\frac{\Pi_{pp}}{B_p}\right)\\ &-\left(\frac{1}{R}\frac{\partial R}{\partial l_p}\right)\Pi_{\phi\phi} - \left(RB_p\frac{\partial(RB_p)^{-1}}{\partial l_p}\right)\Pi_{\psi\psi}\bigg]\\ &+\frac{B_\phi}{R}\bigg[\frac{1}{Rh_p}\frac{\partial}{\partial l_\psi}(R^2h_p\Pi_{\psi\phi})\\ &+B_p\frac{\partial}{\partial l_p}\left(\frac{R\Pi_{p\phi}}{B_p}\right)\bigg].\end{aligned} \tag{16}$$

Comparison with Eq. (6) shows that the second term in Eq. (16) is B_ϕ/R times the toroidal viscous force of Eq. (6). [The second term in Eq. (16) is dominant when $|B_p| < B_\phi|$.] Thus, we see immediately that the toroidal viscous forces caused by the radial transfer of toroidal momentum, which are necessary to explain the rotation measurements, also contribute to the parallel viscous force. This justifies our previous inclusion[9,14] of a term $nmv_d V_\parallel$ in the parallel momentum balance

TABLE II. Comparison of experimental and predicted momentum confinement times.

Experiment	Ion species	Charge	Ion temperature (eV)	Major radius (cm)	Magnetic field (T)	Momentum confinement time experiment	Eq. (15b)
ISX-B	composite	$\overline{Z} = 2.5$	500	93	1.4	16 msec	12 msec
ISX-B	Ne^{IX}	9	800	93	1.4	35 msec	28 msec
PLT	composite	$\overline{Z} = 2.5$	900	130	2.5	10–30 msec	23 msec
PDX	Ti^{XXI}	21	2500	145	2.2	80–100 msec	78 msec

and the assumption $\nu_{d\phi} \equiv \nu_{d\parallel}$ in modeling tokamak plasmas with unbalanced neutral beam injection.

Making use of the viscous stress tensor representation given in Table I and retaining only the leading terms ($f_p \ll f_\phi \simeq 1$), we find the parallel (η_0) viscous force contribution,

$$(\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi})_0 \simeq -2B_p\frac{\partial}{\partial l_p}\left\{\eta_0\left[-\frac{1}{3}\frac{\partial V_p}{\partial l_p} + \left(\frac{1}{R}\frac{\partial R}{\partial l_p}+\frac{1}{3}\frac{1}{B_p}\frac{\partial B_p}{\partial l_p}\right)V_p + f_p R\frac{(V_\phi R^{-1})}{\partial l_p}\right]\right\} + 3\eta_0\left(\frac{\partial B_p}{\partial l_p}\right)\left[-\frac{1}{3}\frac{\partial V_p}{\partial l_p} + \left(\frac{1}{R}\frac{\partial R}{\partial l_p}+\frac{1}{3}\frac{1}{B_p}\frac{\partial B_p}{\partial l_p}\right)V_p + f_p R\frac{\partial(V_\phi R^{-1})}{\partial l_p}\right], \tag{17}$$

the perpendicular ($\eta_1\eta_2$) viscous force contribution,

$$(\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi})_{12} \simeq \frac{B_\phi}{R}\left[\frac{1}{Rh_p}\frac{\partial}{\partial l_\psi}\left(R^3 h_p \eta_2 \frac{\partial(V_\phi R^{-1})}{\partial l_\psi}\right) + B_p\frac{\partial}{\partial l_p}\left(\frac{R^2\eta_2}{B_p}\frac{\partial(V_\phi R^{-1})}{\partial l_p}\right)\right] \equiv (B_\phi/R)(R^2\nabla\phi\cdot\nabla\cdot\mathbf{\Pi})_{12}, \tag{18}$$

and the gyroviscous ($\eta_3\eta_4$) force contribution

$$(\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi})_{34} \simeq \frac{B_\phi}{R}\left[\frac{1}{Rh_p}\frac{\partial}{\partial l_\psi}\left(R^3 h_p \eta_4 \frac{\partial(V_\phi R^{-1})}{\partial l_p}\right) + B_p\frac{\partial}{\partial l_p}\left(\frac{R^2\eta_4}{B_p}\frac{\partial(V_\phi R^{-1})}{\partial l_p}\right)\right] \equiv (B_\phi/R)(R^2\nabla\phi\cdot\nabla\cdot\mathbf{\Pi})_{34}. \tag{19}$$

Making use of the continuity equation $\nabla\cdot n\mathbf{V} = 0$ to write $nV_p = K(\psi)B_p$, Eq. (17) can be used to write the flux surface average of the leading, parallel viscous force contribution as

$$\langle(\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi})_0\rangle = 3\left\langle\eta_0\left(\frac{\partial B_p}{\partial l_p}\right)\left(\frac{1}{R}\frac{\partial R}{\partial l_p}\right)V_p\right\rangle + \left\langle\eta_0\left(\frac{\partial B_p}{\partial l_p}\right)\left(\frac{1}{n}\frac{\partial n}{\partial l_p}\right)V_p\right\rangle + 3\left\langle\eta_0 f_p\left(\frac{\partial B_p}{\partial l_p}\right)R\frac{\partial(V_\phi R^{-1})}{\partial l_p}\right\rangle. \tag{20}$$

Noting that in a low-β equilibrium where RB_p is a surface quantity,

$$\frac{1}{R}\frac{\partial R}{\partial l_p} = -\frac{1}{B_p}\frac{\partial B_p}{\partial l_p},$$

the first term on the rhs of Eq. (20) is of the form given by Hirshman[21] in his derivation of a general constitutive relationship between the parallel viscous force and the flows. The second term arises because of a variation of the particle density over the flux surface, and the third term arises from the departure from rigid rotation over the flux surface; both of these effects are associated with a strongly rotating plasma.

VII. PARTICLE TRANSPORT

The toroidal component of the momentum balance equation for particle species j can be flux-surface-averaged to obtain an expression for the "radial" particle flux across flux surfaces:

$$\Gamma_j \equiv \langle\nabla\psi\cdot n_j\mathbf{V}_j\rangle = -\frac{\langle R^2\nabla\phi\cdot\mathbf{F}_j\rangle}{ej} - \frac{\langle R^2\nabla\phi\cdot\mathbf{M}_j\rangle}{ej} + \frac{\langle R^2\nabla\phi\cdot\nabla\cdot\mathbf{\Pi}_j\rangle}{ej}. \tag{21}$$

The first term is the usual collisional friction-driven transport flux. The second term is the transport flux driven by the momentum input to species j because of collisions with fast injected beam ions. The third term represents explicitly the contribution to the transport flux from the toroidal viscous force, which we have shown to be primarily caused by the gyroviscous effect. We have previously developed[9] a transport theory that includes these last two terms, with the third term represented as a "drag" term $\langle R^2\nabla\Phi\cdot n_j m_j \nu_{dj}\mathbf{V}_j\rangle/e_j$. These previous results can now be taken over by identifying the drag frequency with the gyroviscous term, as given in Eq. (15a).

The combined effect of rotation, beam momentum input, and toroidal viscous force on impurity transport is theoretically predicted[9,16] to enhance the inward transport of impurities with counterinjected neutral beams and to reduce and even reverse the inward transport with co-injected neutral beams. Experimental observations in PLT[22–24] and ISX-B[3] indicate that the central accumulation of edge-injected impurities is substantially increased with counterinjection and reduced with co-injection, in qualitative agreement with the theory.

We further note that the theory[9,16] predicts that the normally negative radial electric field in the center of the plasma is made more negative by counterinjected beams and is made less negative or even positive by counterinjected beams. Measurements[25] in ISX-B qualitatively confirm these trends.

VIII. ROTATION

Solution of the continuity and "radial" momentum balance equations yields[9] leading order expressions for the poloidal and toroidal rotation velocities for species j:

$$V_{jp} = [K_j(\psi)/n_j]B_p \equiv X_j B_p \tag{22}$$

and

$$V_{j\phi} = X_j B_\phi - R(p_j'/n_j e_j + \Phi'), \quad j = 1, ..., J. \tag{23}$$

The surface quantities $K_j(\psi)$ must be determined by using these equations in the flux surface averaged parallel momentum balance equations:

$$\left\langle \mathbf{B}\cdot n_j m_j \frac{\partial \mathbf{V}_j}{\partial t}\right\rangle + \langle \mathbf{B}\cdot n_j m_j (\mathbf{V}_j\cdot\nabla)V_j\rangle + \langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_j\rangle$$
$$= \langle \mathbf{B}\cdot\mathbf{F}_j\rangle + \langle \mathbf{B}\cdot\mathbf{M}_j\rangle,\quad j=1,...,J. \tag{24}$$

In Eqs. (23), Φ is the ambipolar potential, and the prime denotes differentiation with respect to ψ.

Using the results of Sec. VI, we write the viscous force in Eq. (24) in the form

$$\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}\rangle \simeq \langle(\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi})_0\rangle + \langle(\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi})_{34}\rangle$$
$$\simeq \mu X\langle B^2\rangle + \langle \mathbf{B}\cdot nm\nu_d\mathbf{V}\rangle, \tag{25}$$

where the appropriate definition of the "viscosity coefficient" μ follows from the first two terms in Eq. (20) [i.e., it corresponds to Hirshman's[21] μ); we neglect the third term, which is $O(f_p)$].

In order to illustrate the effects, we use a Lorentz model for the collisional friction

$$\mathbf{F}_j = -n_j m_j \sum_{i\neq j}\nu_{ji}(\mathbf{V}_j - \mathbf{V}_i), \tag{26}$$

where ν_{ji} is the Coulomb collision frequency, and make the low-β, circular flux-surface approximation. Using Eqs. (22) and (23), Eqs. (24) can then be reduced to

$$\left\{\frac{\partial X_j}{\partial t} + \left[\left(\frac{\mu_j}{n_j m_j}\right) + \nu_{dj}\right]X_j + \sum_{i\neq j}\nu_{ji}(X_j - X_i)\right\} - \frac{1}{B}\left(\frac{\partial Y}{\partial t} + \nu_{dj}Y\right)$$
$$= \frac{1}{B}\left(\frac{M_j}{n_j m_j} + \nu_{dj}P_j' + \sum_{i\neq j}\nu_{ji}(P_j' - P_i')\right) + O(\epsilon^2),$$
$$j = 1,...,J, \tag{27}$$

where

$$P_j' \equiv \frac{1}{n_j e_j B_p}\frac{\partial p_j}{\partial r},\quad Y \equiv \frac{1}{B_p}\frac{\partial\Phi}{\partial r} = -\frac{E_r}{B_p}. \tag{28}$$

An equation for the evolution of the quantity Y, which is related to the radial electric field, can be obtained by using Eqs. (22) and (23) in the flux surface averaged toroidal momentum balance equations summed over species:

$$\left(\frac{\partial Y}{\partial t} + \hat{\nu}_d Y\right) - \sum_{j=1}^{J}\frac{\rho_j}{\rho}\left(\frac{\partial X_j}{\partial t} + \nu_{dj}X_j\right)B_\phi$$
$$= -\sum_{j=1}^{J}\left[M_{j\phi} + \rho_j\left(\frac{\partial P_j'}{\partial t} + \nu_{dj}P_j'\right)\right], \tag{29}$$

where

$$\rho_j \equiv n_j m_j,\quad \rho \equiv \sum_{j=1}^{J}\rho_j,\quad \hat{\nu}_d \equiv \frac{1}{\rho}\sum_{j=1}^{J}\rho_j\nu_{dj}. \tag{30}$$

It is clear from Eqs. (27) and (29) that the time evolution of the toroidal and poloidal rotation velocities of the various particle species and the time evolution of the radial electric field are coupled. Equations (22) and (23) show that the poloidal and toroidal rotation velocities evolve on the same time scale, that of X_j, rather than separate time scales as previously found[26] in the absence of a gyroviscous term of sufficient magnitude to account for the observed experimental measurements of toroidal rotation.

The elementary time constants of the system of Eqs. (27) and (29) are $(\nu_{dj})^{-1}$, $(\Sigma_j\,\nu_{dj})^{-1}$, and $(\mu_j/n_j m_j + \nu_{dj} + \nu_{ji})^{-1}$. The time constants that determine the evolution of the rotation velocities and the radial electric field involve combinations of the elementary time constants. In particular, the ν_{ij}, which multiply a difference in velocities between species [see Eqs. (26) and (27)] in the governing equations, are suppressed in the composite time constants, which are determined primarily by the ν_{dj} and $\mu_j/n_j m_j$. In the absence of the gyroviscosity, the elementary time constants of the system would be $(\mu_j/n_j m_j + \nu_{ji})^{-1}$. Experimentally ν_{dj} is inferred to be larger than $\mu_j/n_j m_j$ in the collisional regime.

IX. PHYSICAL ORIGIN OF GYROVISCOUS MOMENTUM LOSS

While the physical mechanism of the parallel viscosity tensor $\eta_0\pi^0$ and the perpendicular tensor $\eta_{1,2}\pi^{1,2}$ are transparent and widely described in the literature, the gyroviscous tensor $\eta_{3,4}\pi^{3,4}$ is less understood and deserves discussion here since it would *not* contribute to axial (i.e., z-) momentum loss in a cylindrical tokamak.

The analogy with the collisional heat diffusion problem, recently pointed out by Hogan,[8] is useful here. The terms in the heat flow

$$\mathbf{q} = -\kappa_\parallel\nabla_\parallel T - \kappa_\perp\nabla_\perp T + \kappa\mathbf{b}\times\nabla T, \tag{31}$$

with $\kappa = \frac{5}{2}(nT/eB)$ independent of collision frequency (while $\kappa_\parallel \sim \nu^{-1}$, $\kappa_\perp \sim \nu$) are precisely of the same form as our Eq. (4) for the momentum flow $\boldsymbol{\pi}$. Braginskii[11] points out that (in slab geometry) the last piece in Eq. (31) transports heat not across but *in* the isothermal surface. To obtain a contribution to the cross-field flux $\mathbf{q}\cdot\nabla\psi$, a poloidal variation $(\partial T/\partial l_p)$ [in $T = T_0(\psi) + T_1(\psi,\theta)$] is needed. In a toroidal plasma (tokamak), this gives rise to a Pfirsch–Schlüter factor $1.6\epsilon^2(B_\phi^2/B_p^2)$, thus vanishing for $\epsilon\to 0$. Braginskii[11] shows that the gyroviscous piece of $\boldsymbol{\pi}$ can be basically written in the form

$$\pi^{3,4} = \eta_{3,4}[(\mathbf{b}\times\nabla)\mathbf{V} + \mathbf{V}(\mathbf{b}\times\nabla)], \tag{32}$$

and it is easy to see that it requires $[\partial(V_\phi/R)\partial l_p] = \partial\Omega_1/\partial l_p \neq 0$ {where $\mathbf{V} = R\hat{n}_\phi[\Omega_0(\psi) + \Omega_1(\psi,p)]$} to obtain the dominant contribution to $\boldsymbol{\pi}^{3,4}$. We note from Eq. (32) that (because of $\mathbf{b}\times\nabla$) the gyroviscous stress π_{ik} couples to strain elements $\partial V_m/\partial X_l$ with $l, m\neq i,k$.

Kaufman[12] has provided a microscopic derivation of the tensor element π_{23} which we generalize to the case at hand. Consider an ensemble of ions with random initial conditions $X_i^{(0)}$, random gyrophases α, and random initial velocities $V_\parallel^{(0)}$, but subject to the conditions

$$u_3 \equiv \langle\dot{X}_3\rangle_{X_1,X_2} = u_{3,1}X_1 + u_{3,2}X_2. \tag{33}$$

Here $\langle\ \rangle_{x_1,x_2}$ denotes the ensemble average taken at the point $(X_1,X_2;X_3 = 0)$; $u_{3,1}\equiv\partial u_3/\partial X_1$ and $u_{3,2}$ are given constants and the positions X_i execute the Larmor motion

$$X_1 = X_1^{(0)} - a\cos(\omega t + \alpha),$$
$$X_2 = X_2^{(0)} + a\sin(\omega t + \alpha), \tag{34}$$
$$X_3 = X_3^{(0)} + V_\parallel^{(0)}t.$$

Prescribing the initial conditions $(X_1,X_2;X_3 = 0)$ at $t = 0$ for all particles in the ensemble gives

$$X_1 = X_1^{(0)} - a\cos\alpha, \quad X_2 = X_2^{(0)} + a\sin\alpha, \quad 0 = X_3^{(0)}; \tag{35}$$

then the velocities at $t = 0$ obey

$$\dot{X}_1 = a\omega\sin\alpha = \omega(X_2 - X_2^{(0)}),$$
$$\dot{X}_2 = a\omega\cos\alpha = \omega(X_1^{(0)} - X_1),$$
$$\dot{X}_3 = V_\parallel^{(0)},$$

and we note $\langle \dot{X}_1\rangle = 0 = \langle \dot{X}_2\rangle$, $\langle \dot{X}_3\rangle = \langle V_\parallel^{(0)}\rangle \neq 0$ [see Eq. (33)]. Let the Cartesian coordinates describe a local system at a point on the flux surface such that

$$X_1 \hat{=} X_p, \quad X_2 \hat{=} X_\psi, \quad X_3 \hat{=} X_\phi.$$

The stress tensor element is by definition the ensemble average of the random part of the momentum flow, i.e.,

$$\pi_{23}/mn = \langle (\dot{X}_2 - \langle \dot{X}_2\rangle)(\dot{X}_3 - \langle \dot{X}_3\rangle)\rangle = \langle \dot{X}_2\dot{X}_3\rangle$$
$$= \omega\langle (X_1^{(0)} - X_1)V_\parallel^{(0)}\rangle = \omega\langle X_1^{(0)}V_\parallel^{(0)}\rangle, \tag{36}$$

where we neglected the quadratic terms X_{12}^2, X_1X_2 in the last step. Equation (36) shows that the stress π_{23} originates solely from the correlation between $X_1^{(0)}$ and $V_\parallel^{(0)}$ which follows from the underlying requirement, Eq. (33), as follows. The simplest relation between the random variables $X_1^{(0)}$ and $V_\parallel^{(0)}$ yielding Eq. (33) on ensemble average is $V_\parallel^{(0)} = u_{3,1}X_1^{(0)} + u_{3,2}X_2^{(0)} + \widetilde{V}_\parallel$, where $\widetilde{V}_\parallel$ is an uncorrelated parallel random variable, $\langle \widetilde{V}_\parallel\rangle = 0$. Using this in Eq. (36) yields

$$\pi_{23} = mn\omega\,[\,u_{3,1}(a^2/2 + X_1^2) + u_{3,2}X_1X_2\,]. \tag{37}$$

Since the chosen reference point (X_1, X_2; $X_3 = 0$) can be averaged out over the flux surface, we can set $X_1 = 0 = X_2$ in Eq. (37), and with the gyroradius

$$a = V_{\rm th}/\omega,$$

we obtain

$$\pi_{23} = \frac{mnV_{\rm th}^2}{2\omega}u_{3,1} = \left(\frac{p_\perp}{\omega}\right)\frac{\partial u_3}{\partial X_1},$$

or returning to (ψ,p,ϕ) coordinates

$$\pi_{\psi\phi}^{3,4} = \frac{p_\perp}{\omega}\frac{\partial u_\phi}{\partial l_p}. \tag{38}$$

We see that this viscous stress originates from the $\langle X_p V_\phi\rangle$ correlation ensuing from the preparation $\partial u_\phi/\partial l_p \neq 0$ of the system. Having kept the radial dependence $u_{3,2}X_2 \hat{=} \partial u_\phi/\partial l_\psi$ in Eq. (33) does *not* contribute to the gyroviscous stress $\boldsymbol{\pi}_{\psi\phi}$ and thus the viscous force

$$\hat{n}_\phi\cdot\nabla\cdot\boldsymbol{\pi}^{3,4} = \frac{\partial}{\partial l_\psi}\left(\frac{p_\perp}{\omega}\frac{\partial u_\phi}{\partial l_p}\right) = \frac{\partial}{\partial X_2}\frac{p_\perp}{\omega}u_{3,1} = 0$$

in this model. This is expected for the gyroviscous piece of $\boldsymbol{\pi}$ in analogy to the vanishing cross field heat transport $\kappa\mathbf{b}\times\nabla T$ in slab geometry. The toroidal geometry misaligns the surfaces of constant $V_\phi/R = \Omega(\psi,p)$ with respect to the flux surfaces, leading to the Pfirsch–Schlüter factor $2.3q^2$, where the numerical factor stems from the viscosity coefficient ratio $\eta_4^2/\eta_0\eta_2$ and the finite aspect ratio in the tokamak.[8] There may be other factors which can lead to a larger misalignment (i.e., a larger poloidal variation in V_ϕ/R). We have shown[9] that inertial effects in a strongly rotating ($V_\phi \approx V_{\rm th}$) plasma with impurities can produce $O(\epsilon)$ variations in impurity density, which lead, through poloidal modulation of the radial electric field and other mechanisms, to $O(\epsilon)$ variations in V_ϕ/R, for example.

X. SUMMARY

We have derived the complete viscosity tensor in toroidal geometry for a collisional tokamak plasma with strong rotation ($V_\phi \simeq V_{\rm th}$) and presented a formalism for the calculation of all toroidal and parallel viscous forces in flux surface coordinates. We find that in a plasma with strong rotation, the gyroviscous force that is produced by $O(\epsilon)$ departure from rigid-body rotation within a flux surface is much larger than the perpendicular viscosity and is of sufficient magnitude to account for the momentum confinement times that are inferred from the rotation measurements in PDX, PLT, and ISX. We therefore are able to explain the experimental momentum confinement results without resorting to "anomalous" effects and to justify from first principles our previous use of toroidal and parallel viscous forces of the form $nm\nu_d V_\phi$ and $nm\nu_d V_\parallel$ in the development of a theory for plasma transport and rotation. We have presented simple prescriptions for the drag frequencies, $\nu_d \equiv \tau_\phi^{-1}$, which agree with the experimental results. We show that a viscous force of the magnitude and form that are required to account for the rotation measurements has profound effects upon particle transport and also couples the time evolution of the toroidal and poloidal rotation velocities and the radial electric field to occur on the same time scale.

APPENDIX: DEFINITIONS

The Braginskii decomposition of the viscous stress tensor into parallel (η_0), perpendicular (η_1 and η_2), and gyroviscous (η_3 and η_4) components is given by Eq. (4), where

$$W_{\alpha\beta}^0 \equiv \tfrac{3}{2}(f_\alpha f_\beta - \tfrac{1}{3}\delta_{\alpha\beta})(f_\mu f_\nu - \tfrac{1}{3}\delta_{\mu\nu})W_{\mu\nu},$$
$$W_{\alpha\beta}^1 \equiv (\delta_{\alpha\mu}^\perp\delta_{\beta\nu}^\perp + \tfrac{1}{2}\delta_{\alpha\beta}^\perp f_\mu f_\nu)W_{\mu\nu},$$
$$W_{\alpha\beta}^2 \equiv (\delta_{\alpha\mu}^\perp f_\beta f_\nu + \delta_{\beta\nu}^\perp f_\alpha f_\mu)W_{\mu\nu},$$
$$W_{\alpha\beta}^3 \equiv \tfrac{1}{2}(\delta_{\alpha\mu}^\perp\epsilon_{\beta\gamma\nu} + \delta_{\beta\nu}^\perp\epsilon_{\alpha\gamma\mu})f_\gamma W_{\mu\nu},$$
$$W_{\alpha\beta}^4 \equiv (f_\alpha f_\mu\epsilon_{\beta\gamma\nu} + f_\beta f_\nu\epsilon_{\alpha\gamma\mu})f_\gamma W_{\mu\nu},$$

and

$$\delta_{\alpha\beta}^\perp \equiv \delta_{\alpha\beta} - f_\alpha f_\beta,$$

$\epsilon_{\alpha\beta\gamma}$ is the antisymmetric unit tensor, $f_\alpha \equiv B_\alpha/B$, and the Einstein summation convention is employed. Here $W_{\mu\nu}$ is given in Eq. (3).

The nonvanishing Christoffel symbols for an axisymmetric, tokamak, flux-surface geometry are

$$\Gamma_{\psi p}^\psi = RB_p\frac{\partial(RB_p)^{-1}}{\partial l_p} = -\Gamma_{p\psi}^\psi,$$

$$\Gamma_{\psi p}^p = -\frac{1}{h_p}\frac{\partial h_p}{\partial l_\psi} = -\Gamma_{p\psi}^p,$$

$$\Gamma_{\psi\phi}^\phi = -\frac{1}{R}\frac{\partial R}{\partial l_\psi} = -\Gamma_{\phi\psi}^\phi,$$

$$\Gamma_{p\phi}^\phi = -\frac{1}{R}\frac{\partial R}{\partial l_p} = -\Gamma_{\phi p}^\phi.$$

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