

GEORGIA TECH FUSION RESEARCH CENTER SUMMARY REPORT GTFRC-3  
PARTICLE, MOMENTUM AND ENERGY CONSERVING  
FLUID TRANSPORT THEORY FOR THE TOKAMAK PLASMA EDGE

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**Abstract** A particle-momentum-energy conserving flux-surface-averaged fluid theory for the radial particle and energy fluxes and the radial distributions of pressure, density, rotation velocities and temperatures in the edge plasma is derived from fundamental fluid conservation (particle, momentum, energy) relations. Kinetic corrections arising from ion orbit loss are incorporated into the fluid equations, which are integrated to determine the dependence of the observed edge pedestal profile structure on fueling, heating, electromagnetic and thermodynamic forces. Solution procedures for the fluid plasma and associated neutral transport equations are discussed.

## Outline

- I. Introduction
- II. Ion Orbit Loss
- III. Particle Conservation
- IV. Momentum Conservation
- V. Ion and Electron Energy Conservation
- VI. Ion and Electron Temperatures
- VII. Ion Densities
- VIII. Ohm's Law Radial Electric Field
- IX. Ion Orbit Loss Radial Electric Field
- X. 2D Poloidally Asymmetric Flux Surface Geometry
- XI. Poloidal Asymmetry of Particle and Energy Fluxes in SOL
- XII. Relation to Previous Work
- XIII. A "Pinch-Diffusion" Formulation of the Edge Transport Equations
- XIV. Neutral Particle Transport
- XV. Summary

*A Georgia Tech Fusion Research Center RESEARCH SUMMARY Report.*

## I. Introduction

It is widely believed<sup>1,2</sup>, if not fully understood, that the properties of the high pressure pedestal that characteristically forms over roughly the outer 5-10% of the tokamak plasma in the High Confinement (H-mode) regime (e.g. Refs 3 and 4) discharges influences the performance of the entire plasma.

Recent edge plasma research has been primarily in two areas: i) establishment of the stability boundary in edge physics parameters for the onset of the magneto-hydrodynamic peeling-ballooning mode instability (Edge Localized Mode--ELM) that destroys the edge pedestal (e.g. Ref 5) and thereby *limits the sustainable edge pedestal parameters*; and ii) understanding the balance and transport of particles, momentum and energy in the edge pedestal that *determines* these edge physics parameters--pressure, electric field, rotation, etc. (e.g. Refs 6-9).

This paper i) collects and extends recent research on the individual physical processes involved in plasma edge transport and ii) integrates these individual physical processes into a comprehensive and self-consistent body of fluid transport theory based primarily on the conservation of particles, momentum and energy in the edge pedestal. There are discussions of including kinetic ion orbit loss and intrinsic co-rotation in a fluid theory for the edge pedestal; of determination of the dependence of radial particle and energy fluxes on external and internal sources of particles, momentum, energy and on internal and external forces; of determination of the radial pressure, density and temperature distributions; of determination of the radial electric field and rotation velocities; of different solution methods for the resulting fluid transport equations; of representation of poloidal flux surface geometry asymmetries; and of the poloidal distribution of particle and energy fluxes out of the edge region into the scrape-off layer (SOL), etc.

The primary purpose of this paper is the assemblage in one place of a self-consistent body of theory for plasma transport in the edge (and core) of tokamak plasmas that conserves particles, momentum and energy, and that represent the effect on transport of all important forces acting in the plasma. Calculation models of many of the physical processes involved have been developed in previous papers, and an effort has been made to retain the notation and terminology used in this earlier work to facilitate referral to it.

## II. Ion Orbit Loss

The fluid transport calculation is usually carried out for the distribution of "thermalized" ions and electrons that are flowing radially outward from a central neutral beam, pellet or recycling neutrals ion source in the presence of the distributions of other ion species and electrons. The interactions of these thermalized ion species and the effect on the outward flows of these interactions is traditionally described within the fluid transport theory, making questionable use of particle diffusion theory with "transport coefficients" which are determined from kinetic theory or turbulence theory, or fit to ion density profile measurements. However, there is a further kinetic theory effect associated with those thermalized ions that can access an orbit which carries them across the last closed flux surface (LCFS) and out of the confined plasma.

Although such ion orbit loss (IOL) was identified 20 years ago<sup>10</sup>, IOL has only recently been recognized as an important phenomenon in plasma edge physics<sup>11-17</sup>.

The basic ion orbit calculation is of the minimum energy an ion located at a particular position  $(\psi_0, \theta_0)$  on an internal flux surface with a direction cosine  $\zeta_0$  relative to the toroidal magnetic field direction must have in order to be able to execute an orbit that will cross the loss flux surface (which is taken to be the LCFS or separatrix) at location  $(\psi_{sep}, \theta_{sep})$ . Following Miyamoto<sup>10</sup> and more recently Stacey<sup>13</sup>, we make use of the conservation of canonical toroidal angular momentum

$$RmV_{\parallel}f_{\varphi} + e\psi = const = R_0mV_{\parallel 0}f_{\varphi 0} + e\psi_0 \quad (1)$$

to write the orbit constraint for an ion introduced at a location  $(\psi_0, \theta_0)$  with parallel velocity  $V_{\parallel 0}$ , where  $f_{\varphi} = |B_{\varphi}/B|$ ,  $R$  is the major radius and  $\psi$  is the flux surface value. The conservation of energy and of magnetic moment further require that

$$\begin{aligned} \frac{1}{2}m(V_{\parallel}^2 + V_{\perp}^2) + e\phi = const &= \frac{1}{2}m(V_{\parallel 0}^2 + V_{\perp 0}^2) + e\phi_0 \equiv \frac{1}{2}mV_0^2 + e\phi_0 \\ \frac{mV_{\perp}^2}{2B} = const &= \frac{mV_{\perp 0}^2}{2B_0} \end{aligned} \quad (2)$$

Combining leads to an equation for the minimum speed  $V_0 = \sqrt{V_{\parallel 0}^2 + V_{\perp 0}^2}$  for which an ion at a location  $(r_0, \theta_0)$  on an inner flux surface  $\psi_0$  can reach a location on the separatrix  $\psi$  ( $r_{sep}, \theta_{sep}$ )

$$\begin{aligned} V_0^2 \left[ \left( \left| \frac{B}{B_0} \right| \frac{f_{\varphi 0}}{f_{\varphi}} \zeta_0 \right)^2 - 1 + (1 - \zeta_0^2) \left| \frac{B}{B_0} \right| \right] + V_0 \left[ \frac{2e(\psi_0 - \psi)}{Rmf_{\varphi}} \left( \left| \frac{B}{B_0} \right| \frac{f_{\varphi 0}}{f_{\varphi}} \zeta_0 \right) \right] + \\ \left[ \left( \frac{e(\psi_0 - \psi)}{Rmf_{\varphi}} \right)^2 - \frac{2e(\phi_0 - \phi)}{m} \right] = 0 \end{aligned} \quad (3)$$

where  $\phi$  is the electrostatic potential. The quantity  $\zeta_0 = V_{\parallel 0}/V_0$  is the cosine of the initial guiding center velocity relative to the magnetic field direction. Note that Eq. (3) is quite general with respect to the flux surface geometry representation of  $R$ ,  $B$  and the flux surfaces  $\psi$ .

By specifying an initial "0" location for an ion with initial direction cosine with respect to  $\mathbf{B}$ , denoted  $\zeta_0$ , and specifying a final location on flux surface  $\psi_s$  ( $r_s, \theta_s$ ), Eq. (3) can be solved to determine

the minimum initial ion speed  $V_0$  that is required in order for the ion orbit to reach the separatrix location. Thus, Eq. (3) can be solved for the minimum ion speed or energy necessary for an ion located at some point on an internal flux surface, with a given  $\zeta_0$ , to reach the last closed flux surface  $\psi_{sep}$  (or any other “loss” flux surface) at a given location  $\theta_{sep}$  (or to strike the chamber wall at a given location,  $(\psi_{wall}, \theta_{wall})$ , etc.). A minimum reduced energy for ion orbit loss  $\varepsilon_{min}(\rho, \zeta_0) \equiv 1/2 m V_{0min}^2(\rho, \zeta_0) / k T_{ion}(\rho)$  can thus be calculated for each location on the flux surface designated by  $\rho$ , for each value of the direction cosine<sup>13</sup>.

Such calculations of minimum speeds required for ions with different values of the direction cosine at different poloidal locations on internal flux surfaces to reach the separatrix at various poloidal locations can be carried out straightforwardly<sup>13-15</sup>. Note in this regard that it is the values of the enclosed poloidal flux surfaces (hence the plasma current), and of the electrostatic potential, ion temperature, magnetic field and major radius on these flux surfaces, that are important for the calculation, not the location of the flux surface in Euclidian space per se. We use the fraction of the magnetic flux enclosed to define rho, which should allow an accurate mapping of the experimental parameters to flux surfaces. This result encourages the use of a simple approximate circular flux surface model that conserves flux surface area and enclosed current for the calculations for the general ion orbit loss calculation, the principle geometric approximation of which is that  $RB_\phi = const.$  over the plasma. More realistic flux surface geometries can also be used<sup>16</sup>.

If it is assumed that all ions with speeds greater than or equal to  $V_{0min}$  escape the plasma, the ion orbit loss particle, momentum and energy loss fractions are defined as

$$F_{orb}(\rho) \equiv \frac{N_{loss}}{N_{tot}} = \frac{\int_{-1}^1 \left[ \int_{V_{0min}(\zeta_0)}^{\infty} V_0^2 f(V_0) dV_0 \right] d\zeta_0}{2 \int_0^{\infty} V_0^2 f(V_0) dV_0} = \frac{\int_{-1}^1 \Gamma\left(\frac{3}{2}, \varepsilon_{min}(\rho, \zeta_0)\right) d\zeta_0}{2\Gamma\left(\frac{3}{2}\right)}$$

$$M_{orb}(\rho) \equiv \frac{M_{loss}}{M_{tot}} = \frac{\int_{-1}^1 \left[ \int_{V_{0min}(\zeta_0)}^{\infty} (mV_0\zeta_0) V_0^2 f(V_0) dV_0 \right] d\zeta_0}{2 \int_0^{\infty} (mV_0) V_0^2 f(V_0) dV_0} = \frac{\int_{-1}^1 \zeta_0 \Gamma(2, \varepsilon_{min}(\rho, \zeta_0)) d\zeta_0}{2\Gamma(2)} \quad (4)$$

$$E_{orb}(\rho) \equiv \frac{E_{loss}}{E_{total}} = \frac{\int_{-1}^1 \left[ \int_{V_{0min}(\zeta_0)}^{\infty} \left( \frac{1}{2} m V_0^2 \right) V_0^2 f(V_0) dV_0 \right] d\zeta_0}{\int_{-1}^1 \left[ \int_0^{\infty} \left( \frac{1}{2} m V_0^2 \right) V_0^2 f(V_0) dV_0 \right] d\zeta_0} = \frac{\int_{-1}^1 \Gamma\left(\frac{5}{2}, \varepsilon_{min}(\rho, \zeta_0)\right) d\zeta_0}{2\Gamma\left(\frac{5}{2}\right)}$$

When it is justifiable to use a Maxwellian distribution to evaluate the integrals, the last form obtains, where  $\varepsilon_{min}(\zeta_0) = mV_{0min}^2(\zeta_0)/2kT$  is the reduced energy corresponding to the minimum speed for which ion orbit loss is possible.

From this formulation for the minimum speed for which a particle at a given location on an internal flux surface with a given direction cosine  $\zeta_0$  to develop a numerical procedure<sup>13-15</sup> for calculating the cumulative (with radius) loss fraction for outflowing thermalized ions  $F_j^{iol}(r)$ , for ion energy  $E_j^{iol}(r)$  and for ion toroidal momentum  $M_j^{iol}(r)$  across the LCFS. We estimate<sup>16</sup> from calculation that a fraction  $R_{lossj}^{iol} \approx 0.5$  of these escaping particles do not return into the plasma on the same orbit and thus constitute a kinetic loss of the outflowing fluxes of thermalized ions, their momentum and their energy.

We calculate<sup>8,15</sup> for a variety of DIII-D discharges that the ion orbit loss is negligibly small for normalized radii  $\rho \lesssim 0.9$  but increases rapidly with radius for  $\rho \gtrsim 0.95$ , so that very large fractions of the particles, momentum and energy in the outflowing thermalized ion distribution actually cross the LCFS on IOL orbits which deposit them in the scrape-off layer (SOL) preferentially near the outboard midplane, a phenomenon supported by experimental observation<sup>18-22</sup>. We also find that the preferential loss of counter-current ions leaves an apparent co-current intrinsic rotation of the ions remaining in the edge plasma which peaks about  $\rho \approx 0.97$ , at which location the co-current ions also begin to be ion orbit lost, giving rise to a velocity peaking phenomenon also confirmed by experiment<sup>18-22</sup>.

Thus, taking IOL into account in fluid calculations will greatly affect the calculated distribution both of the radial pressure and temperature profiles of thermalized ions within the edge plasma and the poloidal profiles of thermalized ions, energy and momentum from the edge plasma into the SOL. Our calculation of the ion orbit loss fraction is discussed in Refs 13-17. We have in mind low collisionality edge plasmas and do not include the additional ion orbit loss that would be associated with confined ions scattering into the loss zone in the edge pedestal.

There is also a loss of fast ions produced by neutral beam injection (nbi), which must be taken into account in calculating the source of thermalized ions<sup>16</sup> (and in calculating the alpha particles remaining in the plasma in future fusion plasmas).

### III. Particle Conservation

Particle conservation determines the radial distribution of the thermalized radial ion flux. The flux surface average (FSA) radial particle continuity equation for the main ion species “j” (in the cylindrical approximation) is

$$\begin{aligned} \frac{1}{r} \frac{\partial (r \Gamma_{rj}(r))}{\partial r} &= N_{nbj}(r) (1 - \alpha f_{nbj}^{iol}(r)) + n_e(r) \nu_{ionj}(r) - \alpha \frac{\partial F_j^{iol}(r)}{\partial r} \Gamma_{rj}(r) - \beta z_k \frac{\partial F_k^{iol}}{\partial r} \Gamma_{rk}(r) \\ &\equiv S_{nj} - \alpha \frac{\partial F_j^{iol}(r)}{\partial r} \Gamma_{rj}(r) - \beta z_k \frac{\partial F_k^{iol}}{\partial r} \Gamma_{rk}(r) \end{aligned} \quad (5)$$

where  $N_{nbj}$  and  $f_{nbj}^{iol}$  are the fast neutral beam source rate and the differential (in radius) ion-orbit loss fraction for fast beam ions of species “j”,  $n_e$  is the electron density,  $\nu_{ionj} = n_{oj} \langle \sigma v \rangle_{ionj}$  (with  $n_{oj}$  being the density of neutral atoms of main ion species j,  $\nu_{ionj}$  is the ionization frequency of neutrals of species “j”,  $F_j^{iol}(r)$  is the cumulative ion-orbit loss of thermalized particles of species “j” over  $0 < r' < r$ ,  $\Gamma_{rj}$  is the outward radial particle flux of thermalized main ions of species “j”, and  $\Gamma_{rk}$  is the outward radial particle flux of thermalized “impurity” ions of species “k”, which satisfy an equation similar to (6) but without the neutral beam source term. The parameters  $\alpha$  and  $\beta$  depend on how charge neutrality is assumed to be maintained: i) if neutrality is maintained by a compensating electron loss by unspecified means ( $\alpha = 1, \beta = 0$ ); ii) if neutrality is maintained by a return inward current of thermalized ions from the SOL with a divergence equal to the local IOL ion orbit charge loss rate ( $\alpha = 2, \beta = 1$ ); and iii) if there is no ion orbit loss ( $\alpha = 0, \beta = 0$ ). We think that ( $\alpha = 2, \beta = 1$ ) is the most physically reasonable case.

Equation (5) may be integrated to obtain

$$\begin{aligned} r \Gamma_{rj}(r) &= \int_0^r \left[ N_{nbj} (1 - \alpha f_{nbj}^{iol}) + n_e \nu_{ionj} - \beta z_k \frac{\partial F_k^{iol}}{\partial r} \Gamma_{rk}(r) \right] e^{-\alpha [F_j^{iol}(r) - F_j^{iol}(r')]} r' dr' \\ &\equiv \int_0^r S_{nj}(r') e^{-\alpha [F_j^{iol}(r) - F_j^{iol}(r')]} r' dr' \end{aligned} \quad (6)$$

The radial particle flux of thermalized ions at any radius  $r$  depends on the integral over ion particle flux sources minus sinks at all radii  $0 < r' < r$ , but with sources attenuated by the ion orbit loss of a certain fraction of these source particles. Since the cumulative ion orbit particle loss fraction,  $F_j^{iol}(r)$ , increases sharply with radius just inside the separatrix, this effect would at least partially offset, if not reverse, the effect of the increasing neutrals ionization source just inside the separatrix. A similar result obtains for the impurity ion radial particle flux but involving the impurity edge recycling ion source and ion orbit loss fraction, and without the neutral beam source terms.

Note that no conductive transport coefficient (neoclassical or turbulent) enters the determination of the ion radial particle flux from the particle continuity equation, so the radial particle flux depends on the sources and sinks, not on any diffusive transport coefficient.

#### IV. Momentum Conservation

Radial momentum conservation determines the relation among the radial ion pressure gradient, the radial electric field and the radial distribution of radial  $V \times B$  forces for all species "j". The flux surface average (FSA) of the radial momentum balance for species "j" is

$$\frac{1}{r} \frac{\partial (r p_j)}{\partial r} = n_j e_j E_r + n_j e_j (V_{\theta j} B_\phi - V_{\phi j} B_\theta) \quad (7)$$

Integrating Eq. (7) over  $0 < r < r_{sep}$  yields an expression for the pressure at any location in the edge or core plasma

$$\begin{aligned} r p_j(r) = & r_{sep} p_j(r_{sep}) - e_j \int_r^{r_{sep}} n_j(r') E_r(r') r' dr' + \\ & e_j \int_r^{r_{sep}} n_j(r') V_{\phi j}(r') B_\theta(r') r' dr' - e_j \int_r^{r_{sep}} n_j(r') V_{\theta j}(r') B_\phi(r') r' dr' \end{aligned} \quad (8)$$

where  $r_{sep}$  is the separatrix radius. A similar set of equations with "j" replaced by "k" but without neutral beam sources describes the impurity pressure. Clearly, *the plasma pressure magnitude and distribution depend on the momentum balance among the radial electric field,  $V \times B$  and pressure gradient forces.*

*The toroidal and poloidal fluid rotation velocities are determined largely by the toroidal and poloidal momentum balance equations.* The major computational issue here is the evaluation of the FSA of the toroidal and poloidal components of the viscous stress tensor term  $\langle R \mathbf{n}_\phi \cdot \nabla \cdot \Pi \rangle$ , where  $\Pi$  is the viscous stress tensor. For a toroidally axisymmetric tokamak, the Braginskii rate-of-strain tensor<sup>23</sup>, extended to toroidal flux surface geometry<sup>24</sup> and arbitrary collisionality<sup>25</sup>,  $\Pi_{Brag} = \Pi_{\parallel}(\eta_0) + \Pi_{\Omega}(\eta_{\Omega}) + \Pi_{\perp}(\eta_{\perp})$ , where  $\eta_0 \gg \eta_{\Omega} \gg \eta_{\perp}$  are viscosity coefficients associated with flows parallel to, gyrating about, and perpendicular to the magnetic field, respectively. For toroidally axisymmetric magnetic fields<sup>24,26</sup>  $\langle R \mathbf{n}_\phi \cdot \nabla \cdot \Pi_{\parallel} \rangle \equiv 0$ , leaving the gyroviscous stress as the leading order stress, which may be represented<sup>7</sup> as a "drag" term with a "drag frequency"  $\nu_{\parallel}^{\Omega}$

$$\langle R \mathbf{n}_\phi \cdot \nabla \cdot \Pi_{\Omega} \rangle \simeq n m R \nu_{\parallel}^{\Omega} V_\phi, \quad \nu_{\parallel}^{\Omega} \equiv \frac{1}{2} \eta_{\Omega} \frac{r}{R} (L_n^{-1} + L_T^{-1} + L_{v\phi}^{-1}) \left[ (4 + n_j^c) V_{\phi j}^s + (1 - V_{\phi j}^c) n_j^s \right]. \quad (9)$$

associated with flows encircling the magnetic field, giving rise in an axisymmetric tokamak to gyroviscous stresses, which can be represented in neoclassical theory for axisymmetric tokamaks as set forth in Refs 23-28. The parallel viscosity component of  $\langle R\mathbf{n}_\phi \cdot \nabla \cdot \Pi \rangle$  vanishes identically for an axisymmetric tokamak with zero radial magnetic field<sup>24,26</sup> and the remaining gyro and perpendicular viscosity components can be written in the “drag frequency” form<sup>7</sup>  $\langle R\mathbf{n}_\phi \cdot \nabla \cdot \Pi \rangle \approx nmRv_{vis}^\phi V_\phi$ , where  $v_{vis}^\phi$  is a viscous momentum exchange frequency. (We note there has been some confusion in the literature regarding the gyroviscosity representation of Eq. (5) resulting from two contemporary rotation theories<sup>29,30</sup> that erroneously ordered out  $O(\delta\eta_\Omega)$  terms as higher order in the gyroradius parameter  $\delta \approx 10^{-2}$  relative to  $O(\eta_\perp)$  terms even though  $\eta_\Omega \equiv n_j T_j / \Omega_j \approx 10^4 \eta_\perp$ . However, the correct ordering, plus the relatively good agreement (in the core but not the edge) between direct comparison of calculated and experimental toroidal velocities of Ref. 27 and 28, supports the gyroviscous stress of Eq. (10).)

The toroidal component of the FSA of the inertial force can also be represented as  $\langle R\mathbf{n}_\phi \cdot \nabla nm(\mathbf{V} \cdot \nabla) \mathbf{V} \rangle \approx Rnm(v_{in} + v_{ion})V_\phi$ , where  $v_{in}$  is an “inertial” momentum transport frequency and  $v_{ion}$  is the ionization frequency<sup>7</sup>. The toroidal component of the FSA of the inertial force has the form<sup>7</sup>

$$\begin{aligned} \langle R\mathbf{n}_\phi \cdot \nabla nm(\mathbf{V} \cdot \nabla) \mathbf{V} \rangle &\approx \frac{1}{2} \left[ \frac{V_{rj}}{R} \left\{ \varepsilon \left( 1 + n_j^c + V_{\phi j}^c \right) - 2RL_{V_{\phi j}}^{-1} \right\} - \right. \\ &\left. \varepsilon \frac{V_{\theta j}}{R} \left\{ V_{\phi j}^s \left( 1 + n_j^c + V_{\theta j}^c \right) - V_{\theta j}^s \left( 1 + V_{\phi j}^c \right) - V_{\phi j}^c n_j^s \right\} \right] Rn_j m_j V_{\phi j} \equiv Rnmv_{in} V_\phi \end{aligned} \quad (10)$$

where  $v_{in}$  is an “inertial” momentum transport frequency

Using these representations of the viscous and inertial terms and including also the charge-exchange momentum exchange term, we obtain a composite momentum exchange frequency  $v_d^\phi = v_{vis}^\phi + v_{in} + v_{ion} + v_{cx}$  in terms of which the coupled toroidal momentum balance equations for each ion species can be written in the form

$$n_j m_j (v_{jk} + v_{dj}^\phi) V_{\phi j} - n_j m_j v_{jk} V_{\phi k} = n_j e_j E_\phi^A + e_j B_\theta \Gamma_{rj} + M_{\phi j} \quad (11)$$

with a similar equation (with j and k exchanged) for the impurity ion species “k”. The  $v_{jk}$  is an interspecies collision frequency, the  $v_{dj}^\phi \equiv v_{visj}^\phi + v_{inj} + v_{cxj} + v_{ionj}$  is a composite momentum exchange frequency due to viscosity plus inertia plus charge-exchange plus ionization,  $M_{\phi j}$  is the toroidal momentum source rate (e.g. from neutral beams), and  $E, B$  and  $V$  are the electric and magnetic fields and the fluid velocity.  $E_\phi$  includes the inductive transformer field  $E_\phi^A$  plus any toroidal electric field produced by turbulence.



When these “drag frequency” representations are used for the viscous torque and for the inertial torque, the FSA of the toroidal momentum balance equations for a main ion species “j” and an impurity species “k” can be written

$$\begin{bmatrix} n_j m_j (v_{jk} + v_{dj}^\varphi) & -n_j m_j v_{jk} \\ -n_k m_k v_{kj} & n_k m_k (v_{kj} + v_{dk}^\varphi) \end{bmatrix} \begin{bmatrix} V_{\varphi j} \\ V_{\varphi k} \end{bmatrix} = \begin{bmatrix} n_j e_j \\ n_k e_k \end{bmatrix} E_\varphi^A + \begin{bmatrix} e_j B_\theta & 0 \\ 0 & e_k B_\theta \end{bmatrix} \begin{bmatrix} \Gamma_{rj} \\ \Gamma_{rk} \end{bmatrix} + \begin{bmatrix} M_{\varphi j} \\ M_{\varphi k} \end{bmatrix} \quad (12)$$

where the radial gradient of the velocity has been replaced by a radial gradient scale length of the velocity in the neoclassical  $v_{visc}$ . This replacement is rigorous for the neoclassical rate-of-strain tensor representation of viscosity discussed above (for which the FSA parallel viscosity component vanishes identically for an axisymmetric plasma). Equations (11) and (12) indicate that the toroidal rotation velocities are driven by toroidal electric fields, by radial particle fluxes and by external momentum sources (e.g. neutral beams), with collisional and viscous momentum exchange among the particle species. Extension to treat more than two ion species is straightforward.

We note that such a toroidal viscosity representation is strictly valid only for a toroidally axisymmetric system with no radial magnetic field components, and that the presence of non-axisymmetric radial magnetic field components formally changes the structure of the viscosity representation<sup>26</sup>, as well as introduces a host of new viscosity mechanisms<sup>31,32</sup>. Interestingly, most of these new “neoclassical toroidal viscosity” (NTV) mechanisms are being represented in this “drag frequency” form, so accommodation of the new NVT formalism within the above framework should be straightforward..

A similar derivation can be carried out employing a similar representation for the poloidal component of the viscosity tensor,  $\langle \mathbf{n}_\theta \cdot \nabla \cdot \Pi \rangle \simeq nm v_{vis}^\theta V_\theta$  yielding e.g.

$$\left[ v_{visc\theta j} + v_{jk} + v_{atomj} \right] V_{\theta j} - v_{jk} V_{\theta k} = v_{visj}^\theta \frac{B_\varphi}{B^2} \frac{K^j L_{Tj}^{-1}}{e_j} - \frac{e_j \Gamma_{rj} B_\varphi}{n_j m} \quad (13)$$

and a similar equation with “j” and “k”interchanged<sup>33</sup>. Here,  $v_{atomj} = v_{cxj} + v_{ionj}$ ,  $v_{visj}^\theta = v_{jj}^\theta B \mu_{00} / B_\theta$  and the  $K_j$  and  $\mu_{00}$  are the Hirshman-Sigmar coefficients<sup>33,34</sup>. Combining the equations for the main ions and for the impurity ions yields, in analogy with the above development of the toroidal rotation velocities, but using the extended Hirshman-Sigmar poloidal viscosities<sup>33</sup> for a two-species ion-impurity plasma

$$\begin{aligned}
& \begin{bmatrix} n_j m_j (v_{jk} + v_{dj}^\theta) & -n_j m_j v_{jk} \\ -n_k m_k v_{kj} & n_k m_k (v_{kj} + v_{dk}^\theta) \end{bmatrix} \begin{bmatrix} V_{\theta j} \\ V_{\theta k} \end{bmatrix} = \begin{bmatrix} n_j e_j \\ n_k e_k \end{bmatrix} E_\theta^A - \begin{bmatrix} e_j B_\varphi & 0 \\ 0 & e_k B_\varphi \end{bmatrix} \begin{bmatrix} \Gamma_{rj} \\ \Gamma_{rk} \end{bmatrix} \\
& + \begin{bmatrix} M_{\theta j} \\ M_{\theta k} \end{bmatrix} + \frac{B_\varphi}{B^2} \begin{bmatrix} v_{dj}^\theta K_j / e_j & 0 \\ 0 & v_{dk}^\theta K_k / e_k \end{bmatrix} \begin{bmatrix} L_{Tj}^{-1} \\ L_{Tk}^{-1} \end{bmatrix}
\end{aligned} \tag{14}$$

A similar derivation using the somewhat different Shaing-Sigmar poloidal viscosity representation<sup>25</sup> yields similar results

The fluid velocities calculated from Eqs. (13) and (14) do not include the effects of IOL momentum loss. This can be remedied either by including IOL momentum loss terms in Eqs. (8) and (10) or, equivalently, by considering Eqs. (13) and (14) without IOL momentum loss terms (which are based on fluid momentum balances on the thermalized plasma ions in the absence of IOL) to define *fluid* velocities for the thermalized ions. In the latter case the effect of IOL can be taken into account by calculating an additive intrinsic rotation of these thermalized ions due to the preferential ion orbit loss of counter-current ions. The intrinsic co-current rotation can be written in terms of the cumulative momentum ion orbit loss fraction  $M_j^{iol}$  and the solutions of the fluid Eqs. (13) and (14)

$$V_{\varphi j}^{tot} = V_{\varphi j}^{fluid} + V_{\varphi j}^{intrin} = V_{\varphi j}^{fluid} + \frac{2}{\pi^{1/2}} M_j^{iol} \sqrt{\frac{2kT_j}{m_j}} \tag{15}$$

where  $M_j^{iol}$  is the ion orbit loss fraction of toroidal momentum.

## V. Ion and Electron Energy Conservation

*The ion and electron radial energy fluxes are determined by ion and electron energy conservation.*

The FSA radial energy balance equation for the main ion species is

$$\begin{aligned}
& \frac{1}{r} \frac{\partial (r Q_{rj}(r))}{\partial r} = q_j^{nbi}(r) (1 - \alpha e_{nbi}^{iol}(r)) - q_{je}(r) - q_{jk} - \\
& n_j(r) n_{oj}(r) \langle \sigma v \rangle_{cx} \frac{3}{2} (T(r)_j - T_{oj}) - \frac{\partial E_j^{iol}(r)}{\partial r} Q_{rj}(r)
\end{aligned} \tag{16}$$

with solution

$$rQ_{rj}(r) = \int_0^r \left( q_j^{nbi} (1 - \alpha e_{nbi}^{iol}) - q_{je} - q_{jk} - n_j v_{cxj} \frac{3}{2} (T_j - T_{oj}) \right) e^{-[E_j^{iol}(r) - E_j^{iol}(r')]} r' dr' \quad (17)$$

$$\equiv \int_0^r S_{Ej} e^{-[E_j^{iol}(r) - E_j^{iol}(r')]} r' dr'$$

where  $q_j^{nbi}$  is the neutral beam (or other) heating rate of ion species "j",  $T_j = p_j/n_j$  is the temperature,  $q_{je}$  and  $q_{jk}$  are the energy loss rates due to scattering with electrons and other ion species "k",  $E_j^{iol}$  and  $e_j^{iol}$  are the ion orbit energy loss fractions for thermalized ( $E_j^{iol}$  cumulative) and fast beam ( $e_j^{iol}$  differential) ions of species "j", and the subscript "oj" refers to the neutral atoms of species "j". A calculation<sup>35</sup> for MAST found  $\alpha \simeq 0$  for co-current beam injection, due to heating effects associated with the compensating fast ion current, and  $0.5 < \alpha < 1.0$  for counter-current beam injection. Similar equations obtain for the other ion species "k". Note that the radial ion energy flux at radius  $r$  is determined by sources and sinks of energy and energy ion orbit loss over  $0 < r' < r$ , and not by any conductive or other transport coefficient.

Equation (17) indicates that the radial ion heat flux at radius  $r$  is determined by the net of heat sources and sinks integrated over  $0 < r' < r$  but with the sources attenuated by ion orbit energy loss.

For the electrons the FSA radial energy balance is

$$\frac{1}{r} \frac{\partial (rQ_{re}(r))}{\partial r} = q_e^{nbi}(r) + q_{je}(r) + q_{ke} - n_e(r) n_{oj}(r) \langle \sigma v \rangle_{ionj} E_{ionj} \quad (18)$$

$$- n_e(r) n_{ok}(r) \langle \sigma v \rangle_{ionk} E_{ionk} - n_e n_j L_j(r) - n_e n_k L_k(r)$$

where  $E_{ion}$  is the ionization energy and  $L$  is the radiation emissivity. The last four terms are the ionization energy loss of neutral main and impurity atoms and the radiation loss due to interactions with the main ions and with partially stripped impurity ions, respectively. The solution is

$$rQ_{re}(r) = \int_0^r r' \left[ q_e^{nbi}(r') + q_{je}(r') + q_{ke} - n_e(r') n_{oj}(r') \langle \sigma v \rangle_{ionj} E_{ionj} \right. \quad (19)$$

$$\left. - n_e(r') n_{ok}(r') \langle \sigma v \rangle_{ionk} E_{ionk} - n_e n_j L_j(r') - n_e n_k L_k(r') \right] dr'$$

Thus, conservation of energy specifies the ion and electron FSA radial heat fluxes in terms of the energy sources and sinks for each species, but with ion orbit loss attenuation for the ion energy flux, with no need to use ion or electron thermal diffusion (conduction) coefficients.

## VI. Ion and Electron Temperatures

Among the basic measurements of plasma physics are those of electron densities  $n_e$  and electron  $T_e$  and ion  $T_j$  temperatures and of ion toroidal  $V_{\varphi j}$  and poloidal  $V_{\theta j}$  rotation velocities. On the other hand, the basic conservation equations derived above from the physical principles of conservation of particles, momentum and energy yield solutions for the radial ion particle fluxes  $\Gamma_{rj}$  and  $\Gamma_{rk}$  [Eqs. (5) and (6)]; for the plasma ion pressure  $p = nT$  [Eqs. (7) and (8)]; for the toroidal  $V_{\varphi}$  [Eq. (12)] and poloidal  $V_{\theta}$  [Eq.(14)] ion rotation velocities; and for the total radial energy fluxes of ions [ $Q_{rj}$  Eqs.(16) and (17)] and of electrons [ $Q_{re}$  Eqs.(18) and (19)]. To this we can add the charge neutrality requirement  $n_e = n_j + z^k n_k$ .

Mathematically, the conservation equations that were written down above from physical arguments can be formally derived<sup>36</sup> by taking the first three velocity moments of the Boltzmann transport equation of plasma kinetic theory, and such moments equations are well-known not to be closed, in the sense that no matter how many velocity moments equations are retained, a term involving the next higher moment appears in the last equation, requiring the addition of a closure relation. In the situation we have treated above, the next highest moment that is included in  $Q_{rj}$  and  $Q_{re}$  is the conductive heat flux  $q_{rj}$  and  $q_{re}$ , and the standard procedure<sup>36</sup>, which we follow here, is to use the Fourier heat conduction closure relation  $q = -n\chi\nabla T$  to represent the unknown higher moment in terms of the lower moments  $n$  and  $T$ . The quantity  $Q_j$  is the sum of the conductive and convective heat fluxes and heat fluxes of other forms (rotation, work, viscous) of energy flow. (Since Eqs. (5), (7) and (16) are the zeroth, first and second velocity moments of the Boltzmann transport equation, it would be logical to use the third ( $v^3$ ) velocity moment of the Boltzmann equation to obtain an expression relating the conductive heat flux to the temperature and density, but this quickly becomes computationally intractable and we join everyone else in using the Fourier radial heat conduction relation  $q_r = -\chi n \nabla_r T \equiv \chi n T L_T^{-1}$  as a surrogate closure relation for this purpose to obtain an equation for the ion temperature.)

$$\begin{aligned}
 -n_j \chi_j \left( \frac{1}{r} \frac{\partial (r T_j(r))}{\partial r} \right) &\equiv n_j \chi_j T_j L_{Tj}^{-1} = q_j(r) = Q_j(r) - \frac{5}{2} T_j(r) \Gamma_{rj}(r) - Q_{visj} - Q_{rotj} \\
 &= \frac{1}{r} \int_0^r S_{Ej}(r') e^{-[E_j^{iol}(r) - E_j^{iol}(r')]} r' dr' - \frac{5}{2} T_j(r) \frac{1}{r} \int_0^r S_{nj}(r') e^{-[F_j^{iol}(r) - F_j^{iol}(r')]} r' dr'
 \end{aligned} \tag{20}$$

and for the electron temperature

$$\begin{aligned}
& -n_e \chi_e \left( \frac{1}{r} \frac{\partial (r T_e(r))}{\partial r} \right) \equiv n_e \chi_e T_e L_{Te}^{-1} = q_e(r) = Q_e(r) - \frac{5}{2} T_e(r) \Gamma_{re}(r) \\
& = \frac{1}{r} \int_0^r r' \left[ \begin{aligned} & q_e^{nbi}(r') + q_{je}(r') + q_{ke} - n_e(r') n_{oj}(r') \langle \sigma v \rangle_{ionj} E_{ionj} \\ & - n_e(r') n_{ok}(r') \langle \sigma v \rangle_{ionk} E_{ionk} - n_e n_j L_j(r') - n_e n_k L_k(r') \end{aligned} \right] dr' \\
& \quad - \frac{5}{2} T_e(r) [z_j \Gamma_{rj}(r) + z_k \Gamma_{rk}(r)]
\end{aligned} \tag{21}$$

where  $z_k$  is the charge of the impurity ion. The determination of the total ion or electron radial temperature distributions requires knowledge of the thermal diffusivities due to turbulent or other neoclassical processes, as well as the determination of radial energy fluxes of Eqs. (17) and (19), which latter are sensitive to the ion orbit loss effects.

Once ion and electron temperatures are determined by solving Eqs. (20) and (21) the ion and electron densities can be determined from Eq. (9) and the requirement for charge neutrality  $n_e = n_i + z_k n_k$ .

## VII. Ion Densities

The densities for the main and impurity ions may be determined from pressure balance by dividing Eq. (4) for the main ions by the main ion temperature  $T_j$

$$\begin{aligned}
r n_j(r) = & [r_{sep} p_j(r_{sep}) - e_j \int_r^{r_{sep}} n_j(r') E_r(r') r' dr' + \\
& e_j \int_r^{r_{sep}} n_j(r') V_{\phi j}(r') B_\theta(r') r' dr' - e_j \int_r^{r_{sep}} n_j(r') V_{\theta j}(r') B_\phi(r') r' dr'] / T_j(r)
\end{aligned} \tag{22}$$

and by dividing Eq. (4) for the impurity ions by  $T_k = T_j$

## VIII. Ohm's Law Radial Electric Field

The radial electric field is governed, via momentum conservation constraints, by the ion rotation velocities and pressure gradients. We have already made use of the momentum constraints for the ions, but not yet for the electrons, or equivalently the sum of the momentum constraints over ions and electrons. Multiplying the radial momentum balance Eq. (8) by  $e/m$  and summing over main ions "j", impurity ions "k" and electrons, then using the leading order force balance  $\mathbf{j} \times \mathbf{B} = \nabla p_j + \nabla p_k + \nabla p_e$  leads to a fluid Ohm's Law<sup>37-39</sup> for the determination of

$$E_r^{Ohm} = \eta j_r - \left\{ \left[ \frac{(V_{\theta j} B_\phi - V_{\phi j} B_\theta)}{(1 + n_k m_k / n_j m_j)} \right] + \left[ \frac{(V_{\theta k} B_\phi - V_{\phi k} B_\theta)}{(1 + n_j m_j / n_k m_k)} \right] \right\} - \left\{ \frac{(p_j L_{pj}^{-1} + p_k L_{pk}^{-1})}{e(n_j + z_k n_k)} \right\} \quad (23)$$

Here  $\eta$  is the Spitzer resistivity and the first term is generally small compared to the other two terms. (The second, motional electric field, term and the third, pressure gradient term were dominant in several DIII-D discharges that have been examined<sup>39</sup>.) This expression, when evaluated with rotation velocities determined from experiment, agrees very well<sup>39</sup> in the core plasma with the conventional “experimental” electric field determined by using measured carbon density, temperature and rotation velocity in the radial momentum balance of Eq. (3) for carbon. However, the well-known “wells” (negative peaking) in the  $E_r$  profiles seen in H-mode shots are not predicted by the “fluid” value of the radial electric field given by Eq. (23).

### VIII. Ion Orbit Loss Radial Electric Field

Kinetic ion orbit loss effects produce an additive contribution<sup>40</sup> to the fluid “Ohm’s Law” electrical field of Eq. (23),  $E_r = E_r^{Ohm} + E_r^{iol}$ . The ion orbit loss causes a radially distributed ion charge buildup that must be compensated by an inward ion current, in order to maintain charge neutrality in the plasma, which in turn changes the ion rotation velocities, and the radial electric field must adjust by an amount  $E_r^{iol}$  to maintain the radial ion momentum balance of Eq. (13).

The kinetic loss of ions by IOL constitutes a radially outward kinetic IOL ion current in the plasma edge

$$J_{rIOL}(r) = \frac{1}{r} \int_0^r \left[ e_j N_{nbj}(r') f_{nbj}^{IOL}(r') + e_j \frac{\partial F^{IOLj}(r')}{\partial r'} \Gamma_{rj}(r') + e_k \frac{\partial F^{IOLk}(r')}{\partial r'} \Gamma_{rk}(r') \right] r' dr' \quad (24)$$

Here the first term in the brackets represents the IOL of fast neutral beam ions, and the second and third terms represent the IOL of thermalized ions of the main ion species (j) and of the impurities (k). Charge neutrality of the plasma requires that this kinetic IOL loss current must be compensated.

Neutral atoms from edge recycling or gas fueling are present in the SOL and edge plasma of tokamaks, due to the recycling of ions from the chamber wall and to gas fueling. These neutral atoms are primarily directed inward and transport poloidal moment acquired in the SOL into the edge plasma. The poloidal momentum exchange from neutral atoms to ions in the edge plasma, averaged over a  $0-2\pi$  gyro-orbit, can drive an inward radial current of thermalized ions,  $J_{in} = -J_{IOL}$ , via the poloidal ion momentum balance equation  $\langle (J \times B)_\theta \rangle_{neut} = \langle (J_r B_\phi) \rangle_{neut} = \langle n_i m_i v_{i-n} (V_{n\theta} - V_{i\theta}) \rangle$ <sup>40</sup>. The collision frequency  $\nu_{i-n}$  includes charge-exchange and elastic scattering of ions and neutrals. The  $\langle \rangle$  indicates a  $0-2\pi$  orbit average over the ion gyro-orbit, taking into account variations in ion velocity due to  $E \times B$  and

diamagnetic drifts, and variations in the neutral velocity ( $v_n$ ) direction and density ( $n_n$ ). The basic argument is that when there is an asymmetry in the ion-neutral relative velocity and the neutral density varies around the ion rotation orbit about the magnetic field, net ion motion in the poloidal direction occurs, which generates a radial current. The neutral density increases strongly with increasing radius in the plasma edge. The resulting expression (taking into account ExB and diamagnetic ion drifts and the neutral density variation with radius) for the radial component of the ion current is

$$J_r^{neut} = \frac{n_i m_i v_{in} n_n}{B_\phi} \left( \frac{E_r}{B_\phi} - \frac{\nabla p_i}{e_i B_\phi n_i} + \frac{kT_i \nabla n_n}{e_i B_\phi n_n} \right) \quad (25)$$

Allowing also for an inward viscous current,  $J_r^{visc}$ , and setting  $J_r^{neut} + J_r^{visc} = -J_{rIOL}$  from Eq. (25), Eq. (25) can be solved for the radial electric field required for charge neutrality in the presence of ion orbit loss to provide the inward ion current necessary to compensate the ion orbit loss of ions and maintain charge neutrality

$$\frac{E_r}{B} = \frac{-(J_{rIOL} + J_r^{visc})B}{e_i n_i v_{i-n} n_n} + \frac{\nabla p_i}{e_i n_i} - \frac{kT_i \nabla n_n}{e_i B n_n} \quad (26)$$

All three terms on the right in Eq. (26) are  $< 0$  in the edge plasma of tokamaks, since  $J_{rIOL} > 0$ ,  $\nabla p_i < 0$ ,  $\nabla n_n > 0$ . Ion orbit loss, and hence  $J_{rIOL}$ , are negligible over most of the plasma, but increase dramatically in the outer 10% in radius to just less than 100% of the outflowing ions. The last two terms in Eq. (26) are large just inside the last closed flux surface (LCFS) where the ion pressure gradient is large and negative and the neutral density gradient is large and positive. Both  $J_{rIOL}$  and  $n_n$  are largest just inside the LCFS and decrease inward towards the plasma core. The measured  $E_r^{exp}$  well location agrees with the radial location calculated for the peak of the ion orbit loss distribution in two DIII-D shots<sup>40</sup>, but the details remain to be verified.

## IX. 2D Poloidally Asymmetric Flux Surface Geometry

Although the foregoing equations are based on cylindrical flux surface geometry to minimize the complexity of the formalism, the fluid equations can also be developed on more realistic flux surface geometries, such as the analytical flux surface geometry known as the ‘‘Miller model’’<sup>41</sup> and its various extensions<sup>28,42,43</sup>, or on exact flux surface geometries constructed from experimental measurements of the magnetic field<sup>45</sup>. Comparison calculations<sup>44</sup> indicate that the analytical Miller model flux surfaces can reproduce all features of the exact EFIT<sup>45</sup> flux surfaces.

## X. Poloidal Asymmetry of Particle and Energy Fluxes into SOL

The cumulative (with radius) fraction of thermalized ions and their energy that is being ion orbit lost becomes quite large in the very edge. The poloidal distribution of IOL particles and energy flowing across the separatrix into the SOL is generally calculated to be strongly peaked about the outboard midplane. The implication is that ion orbit loss produces a stronger outboard

peaking of particle and energy fluxes into the SOL than would be predicted by codes that calculate only the diffusive and convective fluid radial particle and heat fluxes.

## XI. Relation to Previous Work

Braginskii<sup>23</sup> and subsequent researchers have taken velocity moments of the Boltzmann transport equation (or approximate versions thereof) to derive fluid particle, momentum and energy balance equations involving various integrals over the distribution function—neoclassical constitutive relations identified in previous sections as viscosity, thermal conductivity, etc. Yet other workers have carried out a similar procedure involving fluctuations to arrive at similar fluid balance equations with turbulent constitutive relations. These fluid balance equations have been combined with the assumptions that particle transport can be described by a diffusion equation and heat conduction is given by the Fourier heat conduction relation to get the modeling equations used in most 1D and 2D plasma fluid codes such as ONE-TWO, SOLPS, UEDGE, EDGE2D-EIRENE, etc.(e.g. Refs 46,47). Such codes typically compensate for omitting the long-range electromagnetic forces in the diffusive particle transport equations as drifts superimposed upon the diffusion theory solutions, if at all. It is not clear that these particle diffusion theory codes conserve particles and momentum (even in the absence of ion orbit loss.)

The energy balance and temperature calculations in this paper are similar to those in these codes, and the rotation velocity calculations are somewhat similar, to the extent they are performed in these codes. However, the use of diffusion theory for the particle balance calculations in these codes is quite different from the particle and momentum conservation approach used in this paper.

The use of diffusion theory to treat particle motion in the presence of significant long range electromagnetic forces is questionable. Diffusion theory was developed for situations in which short-range forces between particles determine a “Fick’s Law” type of diffusive particle flux of the form  $\Gamma = -D\nabla n$ , where  $D$  is a “diffusion constant” characterizing the short-range interactive forces. (Gas molecule/atom diffusion<sup>48</sup> and neutron diffusion<sup>49</sup> are two highly-developed examples of diffusion theory.) When this form of the particle flux is used in the divergence of the particle flux term in the particle balance equation the diffusion theory expression for particle transport  $\nabla \cdot \Gamma = -\nabla \cdot D\nabla n \simeq -D\nabla^2 n$  results. This is not the situation in a plasma with long-range electromagnetic forces.

The kinetic ion orbit loss theory presented in this paper is an important new effect included in the equations of this paper that is not taken into account in any of these codes.

## XII. A “Pinch-Diffusion” Formulation of the Edge Transport Equations

Both because there is a highly developed methodology for solving diffusion equations and because there are several existing plasma transport codes based on diffusion theory, we outline in this section the extension of diffusion theory to incorporate the long-range electromagnetic



forces in the particle and momentum conserving theory of this paper into a pinch-diffusion theory<sup>9,50</sup>.

We have suggested a systematic procedure for developing a momentum-conserving pinch-diffusion theory by substituting the momentum-conserving radial particle flux determined from Eq. (5) into the particle conserving continuity Eq. (1), resulting in a generalized diffusion equation<sup>9,50</sup> (written here in the slab approximation for clarity)

$$-\frac{\partial}{\partial r} \left( D_{jj} \frac{\partial n_j}{\partial r} \right) - \frac{\partial}{\partial r} \left( D_{jk} \frac{\partial n_k}{\partial r} \right) - \frac{\partial}{\partial r} \left( D_{jj} \frac{n_j}{T_j} \frac{\partial T_j}{\partial r} \right) - \frac{\partial}{\partial r} \left( D_{jk} \frac{n_j}{T_k} \frac{\partial T_k}{\partial r} \right) + \frac{\partial (\Gamma_{rj}^{pinch})}{\partial r} = S_j \quad (27)$$

$$D_{jj} \equiv \frac{m_j T_j (v_{dj} + v_{jk})}{(e_j B_\theta)^2}, D_{jk} \equiv \frac{m_j T_j v_{jk}}{(e_j e_k B_\theta^2)}$$

and a similar equation with “j” and “k” interchanged for the impurity species. The  $\Gamma_{rj}^{pinch}$  is a radial “pinch velocity”

$$\Gamma_{rj}^{pinch} = -\frac{n_j E_\phi^A}{B_\theta} - \frac{M_{\phi j}}{e_j B_\theta} + \frac{n_j m_j (v_{jk} + v_{dj})}{e_j B_\theta} \left( \frac{E_r}{B_\theta} + \frac{B_\phi}{B_\theta} V_{\theta j} \right) - n_j m_j v_{jk} V_{\phi k} \quad (28)$$

containing the long range electromagnetic forces and the external momentum input.

Equation (27) can be simplified somewhat by assuming a single impurity species distributed as the main ions with the same temperature. Substituting the resulting expression for  $\Gamma_{rj}$  into the continuity equation results in a somewhat simpler diffusion equation (written now in the cylindrical approximation and taking into account ion orbit loss)

$$-\frac{1}{r} \frac{\partial}{\partial r} \left( D_j \frac{\partial}{\partial r} (r n_j T_j) \right) + \frac{1}{r} \frac{\partial}{\partial r} (r \Gamma_{rj}^{pinch}) + 2 \frac{\partial F_{rj}^{iol}}{\partial r} \left( \Gamma_{rj}^{pinch} - D_j \frac{1}{r} \frac{\partial}{\partial r} (r n_j T_j) \right) = S_{nj} \quad (29)$$

$$D_j \equiv m_j \left( v_{jk} (1 - e_j/e_k) + v_{dj} \right) / (e_j B_\theta)^2$$

It is not clear that this diffusion theory formalism has any computational advantage over direct numerical solution of the previous equations of this paper, and this issue will be investigated.

### XIII. Neutral Particle Transport

The calculation of recycling and fueling neutral fuel atoms and molecules, and of the wall-sputtered or injected impurity atoms are important parts of the edge plasma calculation. Monte Carlo methods are conventionally employed because of the ability to model the complex

geometry of the plasma chamber. However, we have found the TEP interface-current-balance method<sup>51-57</sup> provides comparable accuracy with orders of magnitude less cpu time.

#### **XIV. Summary**

Particle, momentum and energy conserving equations, including kinetic ion orbit loss in the edge plasma, have been used to derive expressions that display explicitly the quantities upon which the pressure, the temperature and the particle density, the radial particle and energy fluxes, and the toroidal and poloidal rotation velocities in the edge plasma depend. The solution of these equations and coupled equations for recycling neutral atoms are discussed, but the development of efficient iterative numerical solution algorithms remains to be worked out. Inclusion in existing codes of ion orbit loss of thermalized ions will rather dramatically change both i) the predicted radial profiles of density, temperature, velocity and radial electric field in the edge pedestal inside the separatrix and ii) the predicted poloidal distribution of ions, energy and momentum from the plasma edge into the scrape-off layer. Most importantly, the equations then used for edge transport calculations will conserve particles, momentum and energy.

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