Convex and Combinatorial Tropical Geometry

Josephine Yu

Abstract

Tropical geometry is the geometry over the max-plus algebra, and it is a degeneration or limit of classical geometric objects under the logarithm or valuation map. In this article we will discuss how to tropicalize algebraic sets, semialgebraic sets, and convex sets, and highlight an application to the truncated moment problem in real algebraic geometry.

1 Tropical algebra and geometry

Tropicalization is a degeneration process that turns an algebraic or semialgebraic set into a polyhedral set. Tropicalization of algebraic varieties, also known as logarithmic limit sets or Maslov dequantization, was introduced by Bergman in [10] to study the "exponential behavior at infinity" of subvarieties of the algebraic torus $(\mathbf{C} \setminus \{0\})^n$. It was used by Viro to construct real plane curves with prescribed degree and topology [35], by Mikhalkin in enumerative algebraic geometry [31], and by Sturmfels for polynomial system solving [34, Chapter 9]. On the more algebraic side, the tropical algebra, or max-plus algebra, has been linked to combinatorial applications such as scheduling and discrete-event dynamical systems [15]. Recently tropical geometry played an important role in the resolution of some long-standing conjectures on complexity of linear programming [3, 4] and log-concavity of coefficients of characteristic polynomials of matroids [1, 7].

The main comprehensive reference on tropical geometry is the text by Maclagan and Sturmfels [29]. The book by Joswig [27] focuses on combinatorial aspects of tropical geometry and applications to optimization, while an earlier book by Itenberg, Mikhalkin, and Shustin focuses on patchworking and enumerative algebraic geometry [22].

The tropicalization of a subset S of $(\mathbf{R} \setminus \{0\})^n$ or $(\mathbf{C} \setminus \{0\})^n$ can be defined as its *logarithmic limit set*

$$\lim_{t \to 0} \log_{\frac{1}{t}} S,$$

consisting of points $y \in \mathbf{R}^n$ for which there is a sequence of points $x^{(t)} \in S$ such that $(\log_{\frac{1}{t}} |x_1^{(t)}|, \ldots, \log_{\frac{1}{t}} |x_n^{(t)}|)$ converges to y as $t \to 0$.

Instead of the parameter t being a real or complex number, we can consider it an element of a field such as the field of Puiseux series in one variable t with real or complex coefficients. A Puiseux series in variable t is a formal power series in $t^{\frac{1}{n}}$ for some natural number n. The complex Puiseux series form an algebraically closed field, while the real Puiseux series form a real closed field.

With the Puiseux series, instead of taking the logarithmic limits, we can take the *valuation* of the series, which is the smallest exponent appearing with a nonzero coefficient. More generally, a *non-Archimedean valuation* on a field $\mathbf{K}$ is a map $\nu : \mathbf{K} \to \mathbf{R} \cup \{\infty\}$ satisfying

1. $\nu(x) = \infty \iff x = 0$

2. $\nu(xy) = \nu(x) + \nu(y)$
3. $\nu(x + y) \geq \min(\nu(x), \nu(y))$.

A valuation is called *trivial* if it sends all of $\mathbf{K} \setminus \{0\}$ to zero. If $\mathbf{K}$ is algebraically closed, then the image of any nontrivial valuation is a dense subset of $\mathbf{R}$.

Tropical geometry arises as the image of usual geometry under the valuation map. For a point $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{K}^n$, we will denote

$$\mathrm{trop}(\mathbf{x}) = (-\nu(x_1), \ldots, -\nu(x_n)).$$

And for a subset $S \subset \mathbf{K}^n$, we can define its tropicalization as

$$\mathrm{trop}(S) = \overline{\{\mathrm{trop}(\mathbf{x}) \mid \mathbf{x} \in S\}}.$$

The tropicalization map turns multiplication into usual addition, and almost turns addition into maximum. We will see now that it is a *hyperfield* homomorphism. The *tropical semiring* or the *max-plus semiring* is the set $\mathbf{R} \cup \{-\infty\}$ with tropical addition $a \oplus b = \max(a, b)$ and tropical multiplication $a \odot b = a + b$.

A hyperfield or hyperring is a set with a multiplication $\odot$ and addition $\boxplus$, where addition may be multivalued, that satisfies a set of axioms similar to those for a field or a ring. Viro pioneered the use of hyperfields in tropical geometry [36], and Baker and Bowler introduced a construction of matroids over hyperfields which unify various matroid concepts such as oriented, valuated, and phased matroids [8].

The tropical hyperfield $\mathbf{T}$ is the set $\mathbf{R} \cup \{-\infty\}$ with multiplication $a \odot b = a + b$ and the addition defined by

$$a \boxplus b = \begin{cases} \max(a, b) \text{ if } a \neq b \\ [-\infty, a] \text{ if } a = b. \end{cases}$$

The tropical addition reflects how addition and valuation interact — we have

$$\begin{aligned} &-\nu(x + y) = \max(-\nu(x), -\nu(y)) \text{ if } \nu(x) \neq \nu(y), \text{ and} \\ &-\nu(x + y) \leq \nu(x) = \nu(y) \text{ otherwise.} \end{aligned}$$

A morphism of hyperfields $\varphi \colon \mathbb{H}_1 \to \mathbb{H}_2$ is a map which induces a homomorphism of multiplicative groups and satisfies $\varphi(x \boxplus y) \subset \varphi(x) \boxplus \varphi(y)$. The negative of the valuation $-\nu$ is a hyperfield homomorphism from $\mathbf{K}$ to the tropical hyperfield $\mathbf{T}$, that is, $-\nu(xy) = -\nu(x) \odot -\nu(y)$ and $-\nu(x + y) \in -\nu(x) \boxplus -\nu(y)$.

2 Tropicalizing algebraic varieties

For a polynomial $f = \sum a_{d_1,\ldots,d_n} x_1^{d_1} \cdots x_n^{d_n} \in \mathbf{K}[x_1, \ldots, x_n]$, let $\mathrm{trop}(f)$ be the polynomial $\boxplus - \nu(a_{d_1,\ldots,d_n}) \odot X_1^{d_1} \odot \cdots \odot X_n^{d_n}$ over the tropical hyperfield obtained by replacing usual operations with tropical operations and each coefficient with its negative valuation. (The powers of the tropical variables X_i's refer to tropical multiplication.)

For example over the field of Puiseux series $\mathbf{K} = \mathbf{C}\{\{t\}\} = \cup_{m \geq 0} \mathbf{C}((t^{1/m}))$ where the valuation is the order of the lowest degree term of the series

$$\mathrm{trop}(1 + (2t^{-3} + 1)x^2 - xy + 5t^{\frac{1}{2}} y^2) = 0 \boxplus (-3 \odot X \odot X) \boxplus (X \odot Y) \boxplus (\frac{1}{2} \odot Y \odot Y).$$

For a point $x \in \mathbf{K}^n$, we have $f(x) = 0 \implies \operatorname{trop}(f)(\operatorname{trop}(x)) \ni -\infty$ since tropicalization is a hyperfield morphism. The only way for a hyperfield sum to contain $-\infty$ is to have maximum attained twice. This means that when evaluating the tropical polynomial trop(f) at the point trop(x), the maximum is attained at least twice. The *tropical hypersurface* of a tropical polynomial F in n variables is defined to be

$$\mathcal{V}(F) = \{\mathbf{x} \in \mathbf{R}^n \mid \text{the maximum is attained at least twice in } F(\mathbf{x})\}.$$

We saw earlier that $f(\mathbf{x}) = 0 \implies \operatorname{trop}(\mathbf{x}) \in \mathcal{V}(\operatorname{trop}(f))$. The following is a converse of this statement.

Theorem 2.1 (Kapranov's Theorem) *For a polynomial f in n variables we have*

$$\mathcal{V}(\operatorname{trop}(f)) = \overline{\{\operatorname{trop}(\mathbf{x}) \mid \mathbf{x} \in \mathbf{K}^n, f(\mathbf{x}) = 0\}}$$

where $\mathbf{K}$ is an algebraically closed extension of the field of definition of f with a non-trivial non-Archimedean valuation, and the closure is in the Euclidean topology.

If f is defined over $\mathbf{C}$, then $\mathbf{K}$ can be taken to be the field of the Puiseux series over $\mathbf{C}$, and the closure is just passing from the rationals to real numbers.

For a polynomial $f = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, we have

$$\operatorname{trop}(f) = \max_{\mathbf{a} \in A} \{\operatorname{trop}(c_{\mathbf{a}}) + (\mathbf{x} \cdot \mathbf{a})\} = \max_{\mathbf{a} \in A} \{(\mathbf{x}, 1) \cdot (\mathbf{a}, \operatorname{trop}(c_{\mathbf{a}}))\}.$$

In particular, when all the coefficients of f have valuation 0, i.e. $\operatorname{trop}(c_{\mathbf{a}}) = 0$ for all $\mathbf{a} \in A$, then $\mathcal{V}(\operatorname{trop}(f))$ consists of vectors $\mathbf{x}$ such that the maximum of $\{\mathbf{x} \cdot \mathbf{a} \mid \mathbf{a} \in A\}$ is attained at least twice. This depends only on the *Newton polytope* of f, which is the convex hull of A. More precisely, the tropical hypersurface $\mathcal{V}(\operatorname{trop}(f))$ is the union of outer normal cones to the edges of the Newton polytope of f. The normal fan of the Newton polytope of f can be recovered from its tropical hypersurface.

If the coefficients of f do not all have the same valuation, then instead of just the Newton polytope, we need to consider the regular subdivision of the Newton polytope induced by the tropicalization of the coefficients.

The fundamental theorem of tropical geometry says that the various ways to tropicalize an algebraic set coincide. For an algebraic set $V \subset \mathbf{C}^n$, let $I(V)$ denote the ideal in $\mathbf{C}[x_1, \ldots, x_n]$ consisting of polynomials vanishing on V, and let $V_{\mathbf{K}}$ denote the set of points in $\mathbf{K}^n$ defined by the same polynomial equations that define V. Proofs for various parts and variations can be found in [33, 26, 32, 29].

Theorem 2.2 (Fundamental Theorem of Tropical Geometry)
Let $V \subset (\mathbf{C} \setminus \{0\})^n$ be an algebraic set. The following subsets of $\mathbf{R}^n$ coincide.

1. *the logarithmic limit set* $\lim_{t \to 0} \log_{\frac{1}{t}} V$
2. $\overline{\{-\nu(x) \mid x \in V_{\mathbf{K}} \cap (\mathbf{K} \setminus \{0\})^n\}}$ *where $\mathbf{K}$ is an algebraically closed field extension of $\mathbf{C}$ with a non-trivial non-Archimedean valuation ν*
3. $\bigcap_{f \in I(V)} \mathcal{V}(\operatorname{trop}(f))$
4. $\{w \in \mathbf{R}^n \mid in_w(I(V))$ *does not contain a monomial* $\}$

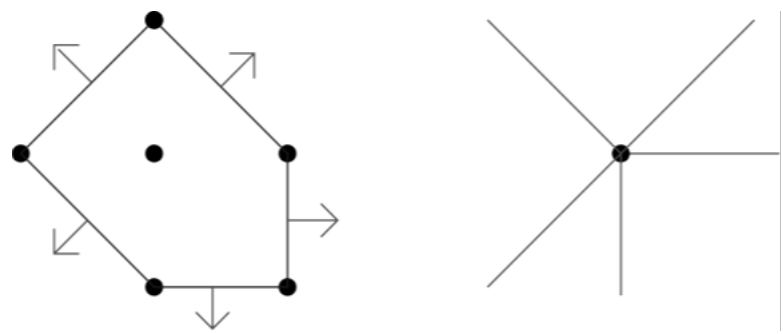

Figure 1: The Newton polygon (left) and the tropical hypersurface (right) of the polynomial $2x + 3x^2 + 5y + 7xy + 11x^2y + 13xy^2$. The coefficients have valuation zero, so they do not affect the picture. The rays of the tropical curve, appropriately weighted, sum to zero.

In (2) the closure is the Euclidean closure. The closure is not needed if the valuation maps $\mathbf{K}$ surjectively onto $\mathbf{R}$ as in the case of generalized Puiseux series where non-rational exponents are allowed. In (4) $in_w(I(V))$ denotes the initial ideal of the ideal $I(V)$ as in Gröbner theory. Kapranov's Theorem is the equivalence of (2) and (3) for the case when the algebraic set V is defined by a single polynomial f. In (3) the intersection can be taken to be finite, and a finite subset of $I(V)$ that does the job is called a *tropical basis*.

For varieties defined over arbitrary algebraically closed fields, the logarithmic limits may no longer make sense, but the equivalence of (2), (3), and (4) still hold. One can also tropicalize using Berkovich analytic spaces; see for example [21].

The set in the theorem is called the tropicalization of V, denoted $\mathrm{trop}(V)$. It is clear from (2) that $\mathrm{trop}(V \cup U) = \mathrm{trop}(V) \cup \mathrm{trop}(U)$ and $\mathrm{trop}(V \cap U) \subseteq \mathrm{trop}(V) \cap \mathrm{trop}(U)$.

Theorem 2.3 (Structure Theorem for Tropical Varieties) *Let V be an irreducible variety of dimension d in $(\mathbf{k} \setminus \{0\})^n$ where $\mathbf{k}$ is an algebraically closed field. Then $\mathrm{trop}(V)$ is a pure d-dimensional balanced polyhedral complex. Moreover it is $d - \ell$ connected through codimension one where ℓ is the dimension of lineality of the polyhedral complex. That is, it is still connected after removing all codimension-two faces and $d - \ell - 1$ codimension-one faces.*

The fact that the tropicalization is a polyhedral complex follows from the characterization (4) of the Fundamental Theorem. The dimension statement is due to Bieri and Groves [11]. The balancing condition, or the zero tension condition is a generalization of the statement that the sum of normal vectors to edges of a polygon is zero, where the vectors are scaled by the lengths of the edges. Connectivity was proven in [14] and [16] and was strengthened in [30] for characteristic zero and in [20] for prime characteristic.

Below we will discuss some examples of tropicalizations. Tropicalization can help uncover combinatorial features of an algebraic variety. But perhaps more importantly tropicalization can lead to some very useful purely combinatorial constructions even when there are no algebraic varieties present.

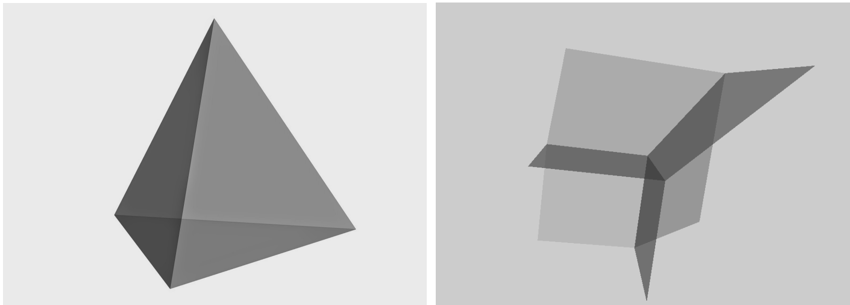

Figure 2: A polytope (left) and the union of normal cones to its edges (right).

When V is a hypersurface defined by a single polynomial f, $\mathrm{trop}(V)$ is the union of normal cones to edges of the Newton polytope of f. This construction of the tropical hypersurface makes sense for arbitrary polytopes. Using the tropical description, some computational problems in polytopes become simpler; for example the tropical hypersurface of the Minkowski sum of two polytopes is just the union of their hypersurfaces. The tropical description is shown to be useful for computing the secondary fan of point configurations [24] and McMullen's polytope algebra [25].

For a linear space L over $\mathbf{C}$, $\mathrm{trop}(L)$ depends only on the matroid of L, that is, the linear independence among columns of a matrix whose rows span L [34, 6]. The tropicalization is called the Bergman fan. The construction of Bergman fans (and more generally tropical linear spaces of valuated matroids) make sense even when the matroid is not representable by linear independence. They are crucial in the work of Adiprasito, Huh, and Katz who proved the conjecture of Rota, Heron, and Welsh on the log-concavity of the coefficients of the chromatic polynomials of matroids, regardless of representability over a field [1, 7]. The Bergman fan is used to develop a Hodge theory that gives combinatorial inequalities. The tropical linear spaces are also essential in the *tropical scheme theory* of Maclagan and Rincón [28].

The tropicalization of an algebraic curve is a graph. The divisor theory on curves "tropicalizes" to chip firing games on graphs and inspires purely combinatorial versions of Riemann–Roch and Abel–Jacobi theory for graphs [9].

3 Tropicalizing semialgebraic sets

A *semialgebraic* subset of $\mathbf{R}^n$ is a set defined by a finite Boolean expression of polynomial inequalities. They can be tropicalized similarly to algebraic sets, using logarithmic limits, valuations, and max-plus algebra. The definition of logarithmic limit sets is exactly the same as before. To use valuations, we need to consider a real closed (instead of algebraically closed) field extension $\mathbf{K}$ with a nontrivial non-Archimedean valuation which is compatible with the ordering. That is, the non-Archimedean absolute value $|x| = e^{-\nu(x)}$ must satisfy

$$0 \leq a \leq b \implies |a| \leq |b|.$$

Concretely we can consider the field of Puiseux series with real coefficients, which is real closed, with ordering given by

$$x > y \text{ if the leading coefficient of } x - y \text{ is positive.}$$

For a polynomial f over $\mathbf{K}$, we can obtain a tropical polynomial $\mathrm{trop}(f)$ as before. For any $\mathbf{x} \in \mathbf{K}^n_{>0}$ and a polynomial f over $\mathbf{K}$, if $f(\mathbf{x}) \geq 0$, then the maximum is attained at a positive term in $\mathrm{trop}(f)\,(\mathrm{trop}(\mathbf{x}))$. Let us define

$$\{\mathrm{trop}(f) \geq 0\} = \{\mathbf{a} \in \mathbf{R}^n \mid \text{ maximum is attained at a positive term in } \mathrm{trop}(f)(\mathbf{a})\}.$$

This is an abuse of notation, since we need to keep track of the signs of the original polynomial f. A neater formulation is to use, as in [23], the tropical real hyperfield introduced in [36]. For example tropicalizing the inequality $x \geq 0$ gives all of $\mathbf{R}$, but tropicalizing $x \leq 1$ gives $\mathbf{R}_{\leq 0}$.

As stated above, we always have

$$\mathrm{trop}\left(\{\mathbf{x} \in \mathbf{K}^n : f(\mathbf{x}) \geq 0\}\right) \subseteq \{\mathrm{trop}(f) \geq 0\}$$

but the containment can be strict, even if the valuation $\nu : \mathbf{K}\backslash\{0\} \to \mathbf{R}$ is surjective. For example, over real Puiseux series $\mathbf{K}$, let $S = \{(x,y) \in \mathbf{K}^2 : (x-2)^2+(y-2)^2 \leq 1\}$. Then $\mathrm{trop}(S)$ consists of a single point $(0,0)$. However $\{\mathrm{trop}(1-(x-2)^2-(y-2)^2) \geq 0\}$ consists of two rays in directions $(0,-1)$ and $(-1,0)$. However, if we use tropicalization of more inequalities satisfied by S, such as $1 \leq x \leq 3, 1 \leq y \leq 3$, then we do obtain exactly $\mathrm{trop}(S)$.

As in the case of algebraic sets, the three ways of tropicalizing coincide for semialgebraic sets. Another way, using analytification, can be found in [23].

Theorem 3.1 (Fundamental Theorem of Real Tropical Geometry) *Let $S \subset \mathbf{R}^n_{>0}$ be a semialgebraic set. Then the following subsets of $\mathbf{R}^n$ coincide.*

1. *the logarithmic limit set $\lim_{t\to 0} \log_{\frac{1}{t}} S$*
2. *$\overline{\{-\nu(\mathbf{x}) \mid \mathbf{x} \in S_{\mathbf{K}}\}}$ where $\mathbf{K}$ is a real closed field extension of $\mathbf{R}$ with a non-trivial non-Archimedean valuation compatible with the order*
3. *$\bigcap_{f\geq 0 \text{ on } S}\{\mathrm{trop}(f) \geq 0\}$*

The descriptions (2) and (3) also work for semialgebraic sets over more general real-closed fields such as the real Puiseux series. The equivalence of (1) and (2) can be found in [2] and (2) and (3) in [23]. In (3) finitely many inequalities suffice. In other words, every semialgebraic set has a finite "tropical basis." Moreover, any tropical polynomial inequality valid on $\mathrm{trop}(S)$ can be "lifted" to a usual polynomial inequality valid on S.

Tropicalizations of semialgebraic sets are closed polyhedral complexes [2, 5] although the dimension of the tropicalization can be strictly smaller than the dimension of the original set, unlike in the case of complex algebraic varieties.

Compared to algebraic varieties, there are fewer known examples of tropicalizations of semialgebraic sets. In fact there is no known algorithms for computing them. Tropicalizations of polytopes over real Puiseux series are tropical polytopes, and they are polyhedral complexes dual to subdivisions of products of simplices [18, 19]. Tropicalization of spectrahedra are related to stochastic mean payoff games, in that feasability problems for the former are equivalent to solving the latter. [5].

4 Tropical Convexity

Tropical operations naturally extend to tropical addition and tropical scalar multiplication on $\mathbf{R}^n$. A subset of $\mathbf{R}^n$ is called *tropically convex* if it is closed under tropical linear combinations. That is, a set $C \subset \mathbf{R}^n$ is tropically convex if for every $\mathbf{x}, \mathbf{y} \in C$ and $a, b \in \mathbf{R}$ we have $(a \odot \mathbf{x}) \oplus (b \odot \mathbf{y}) \in C$. The *tropical convex hull* of a subset of $\mathbf{R}^n$ is the smallest tropically convex set containing it.

A tropical polytope is the tropical convex hull of a finite set of points. Develin and Sturmfels showed that the tropical convex hull of n points in $\mathbf{R}^d$ is dual to a regular subdivision of the product of simplices $\Delta_{n-1} \times \Delta_{d-1}$ induced by the coordinates of these points [18]. The dimension of the tropical convex hull is called the *tropical rank* of the $d \times n$ matrix whose columns are the original points, and it coincides with the largest integer r such that the matrix contains a "non-vanishing" $r \times r$ tropical minor, that is, an $r \times r$ submatrix where the maximum is attained twice in its tropical determinant [17].

Let $\mathbf{K}$ be a real closed field with a non-trivial non-Archimedean valuation compatible to the order, for example, the field of Puiseux series over $\mathbf{R}$ where positive elements are those with positive leading (smallest degree) coefficient. It follows directly from the definitions that the tropicalization of convex cones in $\mathbf{K}^n_{>0}$ are tropically convex, and every tropically convex set arises this way. Note that we do not put non-negativity restrictions on the coefficients a, b in the definition since the tropicalization of $\mathbf{K}_{>0}$ is all of $\mathbf{R}$.

Tropicalization can be used to construct examples of extreme behavior, what happens "at the limit" using the logarithmic limit set point of view. We can take $\mathbf{K}$ to be the field of real Puiseux series (in variable t) convergent on a punctured neighborhood of $0 \in \mathbf{R}$. Then a polytope or a convex set over $\mathbf{K}$ can be viewed as a family of convex sets over $\mathbf{R}$ as $t \to 0$. This was used in [3] to prove that a family of linear programming algorithms — log-barrier interior point methods — are not strongly polynomial. It is still an open question to determine whether there exists a strongly polynomial algorithm for linear programming.

We say that S has the *Hadamard property* if S is closed under coordinate-wise (Hadamard) multiplication. It is known that if S is semialgebraic then $\mathrm{trop}(S)$ is a rational polyhedral complex [2, 5] and if S has Hadamard property then $\mathrm{trop}(S)$ is a closed convex cone, which is equal to the closure of the conical hull of $\log(S)$ [12]. Therefore, if S is a semialgebraic subset of $(\mathbf{R} \setminus \{0\})^n$ with the Hadamard property then $\mathrm{trop}(S)$ is rational polyhedral cone.

For example, the cone of positive semidefinite $n \times n$ matrices has the Hadamard property. Its tropicalization is a polyhedral cone, which is the tropical convex hull of the usual linear space consisting of tropical rank one symmetric matrices [37]. It can be characterized using subdivisions of the twice-dilated simplex $2\Delta_{n-1}$.

Tropicalization of a binomial inequality is a linear inequality, so knowing (the conical hull of) $\mathrm{trop}(S)$ tells us about binomial inequalities on S. This was also used to derive binomial inequalities for graph homomorphism density profiles from extremal graph theory, even when the sets are not semialgebraic [12].

5 Tropical geometry and moment problems

Let $S \subset \mathbf{R}^n$ be a closed semialgebraic set. Usually we take $S = \mathbf{R}^n$ or $S = \mathbf{R}^n_{\geq 0}$. Let A be a finite subset of $\mathbf{N}^n$, considered as a set of monomials. The *moment cone* $M_A(S)$ is the closure of

$$\left\{ \left(\int x^\alpha d\mu \right)_{\alpha \in A} \in \mathbf{R}^A \mid \mu \text{ is a nonnegative Borel measure supported on } S \right\}.$$

The (truncated) moment problem is the membership problem for $M_A(S)$. That is given $L \in \mathbf{R}^A$, determine whether it comes from moments of a nonnegative Borel measure. By linearly extending, L can be considered as a real valued linear function on the space of polynomials with monomials in A. For L to be in the moment cone, it is necessary that L takes nonnegative values on the polynomials which are nonnegative on S. In fact, this necessary condition is also sufficient — the moment cone $M_A(S)$ consists of all linear functions L that are nonnegative on the cone of nonnegative polynomials on S supported on A, that is, $M_A(S)$ is the convex dual of the *nonnegative cone* $Pos_A(S)$ of polynomials which are nonnegative on S. It is generally very difficult to compute (the generators of) the nonnegative cone $Pos_A(S)$.

If S is closed under Hadamard product, then so is $M_A(S)$. We have seen earlier that the tropicalization of such a set is a closed convex cone. Thus $\mathrm{trop}(M_A(S))$ is a closed convex rational polyhedral cone. The linear inequalities valid on $\mathrm{trop}(M_A(S))$ correspond to binomial inequalities on $M_A(S)$. In this way knowing $\mathrm{trop}(M_A(S))$ helps with the moment problem which asks for inequalities defining $M_A(S)$. The following is one of the main results in [13].

Theorem 5.1 *Let $S \subset \mathbf{R}^n_{\geq 0}$ be a semialgebraic set with the Hadamard property such that the intersection with the positive orthant is dense in S. The tropicalization of the truncated moment cone $M_A(S)$ is the rational polyhedral cone of functions $h\colon A \to \mathbf{R}$ satisfying the following linear inequalities:*

1. *(Convexity) $\sum_{i=1}^r \lambda_i h(a_i) \geq h(b)$ for all $a_1, \ldots, a_r, b \in A$, $\lambda_i \geq 0$, $\sum_{i=1}^r \lambda_i = 1$;*
2. *(Non-increasing) $h(a) \geq h(b)$ whenever $a - b \in \mathrm{trop}(S)^\vee$.*

Here $\mathrm{trop}(S)$ is a rational polyhedral cone, and $\mathrm{trop}(S)^\vee$ denotes its convex dual cone. For example, when $S = \mathbf{R}^n_{\geq 0}$, we have $\mathrm{trop}(S) = \mathbf{R}^n$, so the condition (2) is vacuous. When S is the unit cube $[0,1]^n$, we have $\mathrm{trop}(S) = \mathbf{R}^n_{\leq 0}$, so the condition (2) says that the function h is decreasing in coordinate directions.

Instead of the cone of nonnegative polynomials, it is often easier to understand the cone of sum-of-square (SOS) polynomials, and semidefinite programming can be used to do computations with them. The dual cone of SOS polynomials is often called the *pseudo-moment cone*. An important question is to characterize for which S and A the moment and pseudo-moment cones coincide.

When S is not all of $\mathbf{R}^n$, there can be different choices on what we mean by "sum of squares". If S is the positive orthant, then all monomials are nonnegative on S, thus we allow monomial multipliers when taking the sums of squares. Similarly, when $S = [0,1]^n$ we allow the $(1 - x_i)$ termss to be multiplied by the squares. In general, being "sum of squares" may depend not only on S but also on the choice of semialgebraic description of S.

We can describe tropicalizations of pseudo-moment cones similarly to the above theorem, where the convexity condition is replaced by the following *midpoint convexity* condition: $h(a_1) + h(a_2) \geq 2h(b)$ for all $a_1, a_2, b \in A$ with $a_1 + a_2 = 2b$.

The first known polynomial that is nonnegative on $\mathbf{R}^2$ but is not SOS is the Motzkin polynomial $1 + x^2y^4 + x^4y^2 - 3x^2y^2$. The monomial support A in this case is

$$A = \{(0,0), (2,4), (4,2), (2,2)\}.$$

The midpoint convexity condition is vacuous on A, but there is a convexity condition $m_{(0,0)} + m_{(2,4)} + m_{(4,2)} \geq 3m_{(2,2)}$, showing that there is a nonnegative polynomial on this support which is not SOS.

Acknowledgements

The author was supported in part by United States National Science Foundation grant #1855726. The author thanks the anonymous referees for helpful comments and May Cai for pointing out some typographical errors.

References

[1] Karim Adiprasito, June Huh, and Eric Katz, *Hodge theory for combinatorial geometries*, Ann. of Math. (2) **188** (2018), no. 2, 381–452. MR 3862944

[2] Daniele Alessandrini, *Logarithmic limit sets of real semi-algebraic sets*, Adv. Geom. **13** (2013), no. 1, 155–190. MR 3011539

[3] Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, and Michael Joswig, *Log-barrier interior point methods are not strongly polynomial*, SIAM J. Appl. Algebra Geom. **2** (2018), no. 1, 140–178. MR 3772004

[4] ______, *What tropical geometry tells us about the complexity of linear programming*, SIAM Rev. **63** (2021), no. 1, 123–164. MR 4209657

[5] Xavier Allamigeon, Stéphane Gaubert, and Mateusz Skomra, *Tropical spectrahedra*, Discrete Comput. Geom. **63** (2020), no. 3, 507–548. MR 4074332

[6] Federico Ardila and Caroline J. Klivans, *The Bergman complex of a matroid and phylogenetic trees*, J. Combin. Theory Ser. B **96** (2006), no. 1, 38–49. MR 2185977

[7] Matthew Baker, *Hodge theory in combinatorics*, Bull. Amer. Math. Soc. (N.S.) **55** (2018), no. 1, 57–80. MR 3737210

[8] Matthew Baker and Nathan Bowler, *Matroids over partial hyperstructures*, Adv. Math. **343** (2019), 821–863. MR 3891757

[9] Matthew Baker and Serguei Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*, Adv. Math. **215** (2007), no. 2, 766–788. MR 2355607

[10] George M. Bergman, *The logarithmic limit-set of an algebraic variety*, Trans. Amer. Math. Soc. **157** (1971), 459–469. MR 280489

[11] Robert Bieri and J. R. J. Groves, *The geometry of the set of characters induced by valuations*, J. Reine Angew. Math. **347** (1984), 168–195. MR 733052

[12] Grigoriy Blekherman, Annie Raymond, Mohit Singh, and Rekha R. Thomas, *Tropicalization of graph profiles*, arXiv: Combinatorics (2020).

[13] Grigoriy Blekherman, Felipe Rincón, Rainer Sinn, Cynthia Vinzant, and Josephine Yu, *Moments, sums of squares, and tropicalization*, preprint (2021).

[14] T. Bogart, A. N. Jensen, D. Speyer, B. Sturmfels, and R. R. Thomas, *Computing tropical varieties*, J. Symbolic Comput. **42** (2007), no. 1-2, 54–73. MR 2284285

[15] Peter Butkovič, *Max-algebra: the linear algebra of combinatorics?*, Linear Algebra Appl. **367** (2003), 313–335. MR 1976928

[16] Dustin Cartwright and Sam Payne, *Connectivity of tropicalizations*, Math. Res. Lett. **19** (2012), no. 5, 1089–1095. MR 3039832

[17] Mike Develin, Francisco Santos, and Bernd Sturmfels, *On the rank of a tropical matrix*, Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ., vol. 52, Cambridge Univ. Press, Cambridge, 2005, pp. 213–242. MR 2178322

[18] Mike Develin and Bernd Sturmfels, *Tropical convexity*, Doc. Math. **9** (2004), 1–27. MR 2054977

[19] Mike Develin and Josephine Yu, *Tropical polytopes and cellular resolutions*, Experiment. Math. **16** (2007), no. 3, 277–291. MR 2367318

[20] Francesca Gandini, Milena Hering, Diane Maclagan, Fatemeh Mohammadi, Jenna Rajchgot, Ashley K. Wheeler, and Josephine Yu, *Toric and tropical bertini theorems in prime characteristic*, preprint arXiv:2111.13214 (2021).

[21] Walter Gubler, *A guide to tropicalizations*, Algebraic and combinatorial aspects of tropical geometry, Contemp. Math., vol. 589, Amer. Math. Soc., Providence, RI, 2013, pp. 125–189. MR 3088913

[22] Ilia Itenberg, Grigory Mikhalkin, and Eugenii Shustin, *Tropical algebraic geometry*, Oberwolfach Seminars, vol. 35, Birkhäuser Verlag, Basel, 2007. MR 2292729

[23] Philipp Jell, Claus Scheiderer, and Josephine Yu, *Real tropicalization and analytification of semialgebraic sets*, International Mathematics Research Notices (2020).

[24] Anders Jensen and Josephine Yu, *Computing tropical resultants*, J. Algebra **387** (2013), 287–319. MR 3056698

[25] ———, *Stable intersections of tropical varieties*, J. Algebraic Combin. **43** (2016), no. 1, 101–128. MR 3439302

[26] Anders Nedergaard Jensen, Hannah Markwig, and Thomas Markwig, *An algorithm for lifting points in a tropical variety*, Collect. Math. **59** (2008), no. 2, 129–165. MR 2414142

[27] Michael Joswig, *Essentials of tropical combinatorics*, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2022.

[28] Diane Maclagan and Felipe Rincón, *Tropical schemes, tropical cycles, and valuated matroids*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 3, 777–796. MR 4055988

[29] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015. MR 3287221

[30] Diane Maclagan and Josephine Yu, *Higher connectivity of tropicalizations*, Mathematische Annalen (2021).

[31] Grigory Mikhalkin, *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. **18** (2005), no. 2, 313–377. MR 2137980

[32] Sam Payne, *Fibers of tropicalization*, Math. Z. **262** (2009), no. 2, 301–311. MR 2504879

[33] David Speyer and Bernd Sturmfels, *The tropical Grassmannian*, Adv. Geom. **4** (2004), no. 3, 389–411. MR 2071813

[34] Bernd Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conference Series in Mathematics, vol. 97, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2002. MR 1925796

[35] Oleg Viro, *Dequantization of real algebraic geometry on logarithmic paper*, European Congress of Mathematics, Vol. I (Barcelona, 2000), Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 135–146. MR 1905317

[36] Oleg Viro, *Hyperfields for tropical geometry I. Hyperfields and dequantization*, arXiv:1006.3034, 2010.

[37] Josephine Yu, *Tropicalizing the positive semidefinite cone*, Proc. Amer. Math. Soc. **143** (2015), no. 5, 1891–1895. MR 3314099