

ECE-3075A Random Signals
Summer 2003 - Final Exam

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Problem #1 (7pts ea., 21pts total)

A random variable X has a probability density function of the form $f_X(x) = \exp\{-a|x|\}$ $-\infty < x < \infty$

A second random variable Y is related to X by $Y = |X|$.

- Find the numeric value of a ;
- Find the probability density function of Y ;
- A third random variable Z is a function of X and Y , i.e., $Z = \frac{X}{Y}$; find the first and second moment of Z .

a) $f_X(x) = \exp\{-a|x|\}$ $-\infty < x < \infty$

$$\int_{-\infty}^{\infty} \exp\{-a|x|\} dx = 2 \int_0^{\infty} \exp\{-ax\} dx = \frac{2}{a} = 1 \quad \text{therefore, } a = 2$$

b) $Y = |X| \quad \frac{dx}{dy} = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases} \quad f_X(x) = \exp\{-a|x|\} \quad -\infty < x < \infty$

$$f_Y(y) = (|1|) \exp\{-ay\} + (|-1|) \exp\{-ay\} = 2 \exp\{-2y\} \quad 0 \leq y < \infty$$

c) $Z = \frac{X}{Y} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

Since $f_X(x) = \exp\{-a|x|\}$, $-\infty < x < \infty$, is symmetric with respect to 0, $\Pr\{Z = 1\} = \Pr\{Z = -1\} = 0.5$

$$\bar{Z} = E\{Z\} = -1 \times 0.5 + 1 \times 0.5 = 0$$

$$\overline{Z^2} = E\{Z^2\} = 1^2 \times 0.5 + (-1)^2 \times 0.5 = 1$$

Problem #2 (7pts ea., 21pts total)

The characteristic function of two independent Gaussian random variable, X and Y , are given by $\Phi_X(u) = e^{ju3-(2u^2)}$ and $\Phi_Y(v) = e^{jv-(4v^2)}$, respectively. Let $Z = X - Y$.

- Find $\bar{Z} = E[Z]$;
- Find $\overline{Z^2} = E[Z^2]$;
- Find the characteristic function of Z , $\Phi_Z(w)$.

$$\Phi_X(u) = e^{ju3-(2u^2)} = e^{ju\bar{X}-(\sigma_X^2 u^2/2)} \quad \therefore \quad \bar{X} = 3, \quad \sigma_X^2 = 4, \quad \text{and} \quad \overline{X^2} = \sigma_X^2 + \bar{X}^2 = 13$$

$$\Phi_Y(v) = e^{jv-(4v^2)} = e^{jv\bar{Y}-(\sigma_Y^2 v^2/2)} \quad \therefore \quad \bar{Y} = 1, \quad \sigma_Y^2 = 8, \quad \text{and} \quad \overline{Y^2} = \sigma_Y^2 + \bar{Y}^2 = 9$$

a) $\bar{Z} = E[Z] = E[X - Y] = E[X] - E[Y] = 3 - 1 = 2$

b) $\overline{Z^2} = E[Z^2] = E[(X - Y)^2] = E[X^2] + E[Y^2] - 2E[X]E[Y] = 13 + 9 - 2 \times 3 \times 1 = 22 - 6 = 16$

c) $E[-Y] = -E[Y] = -1$, $E[Y^2] = E[(-Y)^2] = 9$, and $\sigma_{-Y}^2 = 9 - (-1)^2 = 8 = \sigma_Y^2$

$$\Phi_{-Y}(v) = e^{jv(-\bar{Y})-(\sigma_Y^2 v^2/2)} = e^{-jv-(4v^2)}$$

$$Z = X - Y = X + (-Y) \quad f_Z(z) = f_X(z) * f_{-Y}(z)$$

$$\text{Therefore, } \Phi_Z(w) = \Phi_X(w)\Phi_{-Y}(w) = e^{jw3-(2w^2)} e^{-jw-(4w^2)} = e^{jw2-(6w^2)}$$

Indeed, $\sigma_Z^2 = \overline{Z^2} - (\bar{Z})^2 = 16 - 2^2 = 12$ and $(\sigma_Z^2/2) = 6$ which appears in the characteristic function.

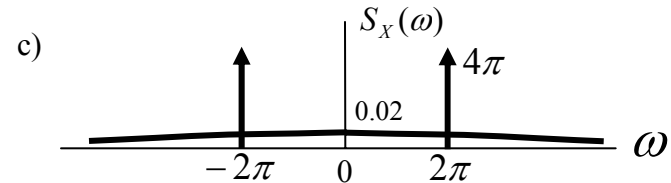
Problem #3 (7 pts ea., 42 pts total)

Consider a signal/process $X(t) = A \cos(2\pi t + \Theta) + V(t)$ where A is a Gaussian random variable with zero mean and a variance of 8, Θ is uniformly distributed in $(0, 2\pi)$ and the independent noise $V(t)$ has an autocorrelation function of $R_V(\tau) = 10e^{-1000|\tau|}$. The signal is applied to a finite-time integrator with impulse response $h(t) = [u(t) - u(t-2)]/2$.

- Find the autocorrelation of the input $R_X(\tau)$;
- Find the spectral density of the input $S_X(\omega)$;
- Sketch the input spectral density;
- Find the autocorrelation of the output $R_Y(\tau)$;
- Find the spectral density of the output $S_Y(\omega)$;
- Sketch the output spectral density.

$$\begin{aligned} \text{a) } R_X(\tau) &= E[X(t)X(t+\tau)] = E\{[A \cos(2\pi t + \Theta) + V(t)][A \cos(2\pi t + 2\pi\tau + \Theta) + V(t+\tau)]\} \\ &= E\{A \cos(2\pi t + \Theta)A \cos(2\pi t + 2\pi\tau + \Theta)\} + R_V(\tau) = \frac{1}{2}E\{A^2\} \cos 2\pi\tau + R_V(\tau) = 4 \cos 2\pi\tau + 10e^{-1000|\tau|} \end{aligned}$$

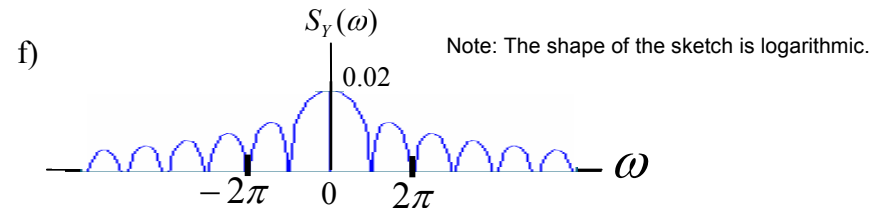
$$\begin{aligned} \text{b) } S_X(\omega) &= \mathbf{F}\{R_X(\tau)\} = \mathbf{F}\{4 \cos 2\pi\tau\} + \mathbf{F}\{R_V(\tau)\} \\ &= 4\pi\{\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\} + \frac{2 \times 10^4}{10^6 + \omega^2} \end{aligned}$$



$$\begin{aligned} |H(\omega)|^2 &= \text{sinc}^2\left(\frac{\omega}{\pi}\right) & Y(t) &= \int_{-\infty}^{\infty} X(t-\lambda)h(\lambda)d\lambda = \frac{1}{2} \int_0^2 [A \cos(2\pi t - 2\pi\lambda + \Theta) + V(t-\lambda)]d\lambda \\ & & &= \frac{1}{20\pi} A \sin(2\pi t - 2\pi\lambda + \Theta) \Big|_{\lambda=0}^{\lambda=2} + \left\{ \frac{1}{2} \int_0^2 V(t-\lambda)d\lambda \right\} = \frac{1}{2} \int_0^2 V(t-\lambda)d\lambda = V(t) * h(t) \end{aligned}$$

$$\text{e) } S_Y(\omega) = S_X(\omega) |H(\omega)|^2 = \text{sinc}^2\left(\frac{\omega}{\pi}\right) \left\{ 4\pi\{\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\} + \frac{2 \times 10^4}{10^6 + \omega^2} \right\} = \text{sinc}^2\left(\frac{\omega}{\pi}\right) \frac{2 \times 10^4}{10^6 + \omega^2} \approx 0.02 \times \text{sinc}^2\left(\frac{\omega}{\pi}\right)$$

$$\begin{aligned} \text{d) } R_Y(\tau) &= \mathbf{F}^{-1}\{S_Y(\omega)\} \approx \mathbf{F}^{-1}\left\{0.02 \text{sinc}^2\left(\frac{\omega}{\pi}\right)\right\} \\ &= 0.02 \times \frac{1}{2} \left(1 - \frac{|\tau|}{2}\right) = 0.01 \left(1 - \frac{|\tau|}{2}\right), \quad |\tau| < 2 \end{aligned}$$



Problem #4 (7 pts ea., 21 pts total)

A random process having an autocorrelation of $R_X(\tau) = 16e^{-2|\tau|} + 16$ is the input to a linear system having an impulse response of $h(t) = \delta(t) - 2e^{-2t}u(t)$

- Find the mean value of the output;
- Find the mean-square value of the output;
- Find the variance of the output.

a) $R_X(\tau) = 16e^{-2|\tau|} + 16$ The constant component in $R_X(\tau)$ corresponds to $(\bar{X})^2$. Therefore, $\bar{X} = 4$.

$$\bar{Y} = \bar{X} \int_{-\infty}^{\infty} h(t) dt = 4 \int_{-\infty}^{\infty} [\delta(t) - 2e^{-2t}] dt = 4[1 - 1] = 0$$

$$\begin{aligned} \text{b) } \overline{Y^2} &= R_Y(0) = \int_0^{\infty} d\lambda_1 \int_0^{\infty} R_X(\lambda_2 - \lambda_1) h(\lambda_1) h(\lambda_2) d\lambda_2 = \int_0^{\infty} h(\lambda_2) d\lambda_2 \int_0^{\infty} R_X(\lambda_2 - \lambda_1) h(\lambda_1) d\lambda_1 \\ &= \int_0^{\infty} R_X(\lambda_2 - \lambda_1) h(\lambda_1) d\lambda_1 = \int_0^{\infty} [16e^{-2|\lambda_2 - \lambda_1|} + 16][\delta(\lambda_1) - 2e^{-2\lambda_1}] d\lambda_1 = \int_0^{\infty} 16e^{-2|\lambda_2 - \lambda_1|} [\delta(\lambda_1) - 2e^{-2\lambda_1}] d\lambda_1 = 16e^{-2|\lambda_2|} - \int_0^{\infty} [32e^{-2|\lambda_2 - \lambda_1|} e^{-2\lambda_1}] d\lambda_1 \\ &= \int_0^{\infty} [32e^{-2|\lambda_2 - \lambda_1|} e^{-2\lambda_1}] d\lambda_1 = \int_0^{\lambda_2} [32e^{-2(\lambda_2 - \lambda_1)} e^{-2\lambda_1}] d\lambda_1 + \int_{\lambda_2}^{\infty} [32e^{-2(\lambda_2 + \lambda_1)} e^{-2\lambda_1}] d\lambda_1 = 8e^{-2\lambda_2} + 32\lambda_2 e^{-2\lambda_2} \end{aligned}$$

$$\int_0^{\infty} R_X(\lambda_2 - \lambda_1) h(\lambda_1) d\lambda_1 = 8e^{-2\lambda_2} - 32\lambda_2 e^{-2\lambda_2} \quad \text{for } \lambda_2 \geq 0$$

$$\overline{Y^2} = \int_0^{\infty} [8e^{-2\lambda_2} - 32\lambda_2 e^{-2\lambda_2}][\delta(\lambda_2) - 2e^{-2\lambda_2}] d\lambda_2 = 8 - \int_0^{\infty} [16e^{-4\lambda_2} - 64\lambda_2 e^{-4\lambda_2}] d\lambda_2 = 8 - 4 + \int_0^{\infty} 64\lambda_2 e^{-4\lambda_2} d\lambda_2 = 4 + 4 = 8$$

Alternatively,

$$S_X(\omega) = \mathbf{F}\{R_X(\omega)\} = 16 \times 2\pi\delta(\omega) + 16 \frac{4}{4 + \omega^2} = 32\pi\delta(\omega) + \frac{64}{4 + \omega^2}$$

$$\mathbf{F}\{h(t)\} = \mathbf{F}\{\delta(t) - 2e^{-2t}u(t)\} = 1 - \frac{2}{2 + j\omega} = \frac{j\omega}{2 + j\omega} = H(\omega) \quad \therefore |H(\omega)|^2 = \frac{\omega^2}{4 + \omega^2}$$

$$S_Y(\omega) = S_X(\omega) |H(\omega)|^2 = \left[32\pi\delta(\omega) + \frac{64}{4 + \omega^2} \right] \frac{\omega^2}{4 + \omega^2} = \frac{64\omega^2}{(4 + \omega^2)^2}$$

$$\overline{Y^2} = R_Y(0) = \frac{1}{\pi} \int_0^{\infty} \frac{64\omega^2}{(4 + \omega^2)^2} d\omega = \frac{64}{\pi} \left[\frac{-\omega}{2(4 + \omega^2)} + \frac{1}{4} \tan^{-1}\left(\frac{\omega}{2}\right) \right]_0^{\infty} = \frac{64}{\pi} \frac{\pi}{8} = 8$$

$$\text{c) } \sigma_Y^2 = \overline{Y^2} - (\bar{Y})^2 = 8$$