ECE 3075A
Random Signals

Lecture 10

Multivariate Gaussian Distribution, and Central Limit Theorem

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Characteristic Function of A Gaussian R.V.

$$\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{jux} f_X(x)\,dx$$

$$\Phi_X(u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(jux) \exp\left[\frac{-(x-\overline{X})^2}{2\sigma^2}\right] dx$$

The exponential term:

$$jux - \frac{(x-\overline{X})^2}{2\sigma^2} = \frac{jux\,2\sigma^2 - (x-\overline{X})^2}{2\sigma^2} = -\frac{\left[x-(\overline{X}+ju\sigma^2)\right]^2}{2\sigma^2} + ju\overline{X} - \frac{u^2\sigma^2}{2}$$

$$\Phi_X(u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left[x-(\overline{X}+ju\sigma^2)\right]^2}{2\sigma^2} + ju\overline{X} - \frac{u^2\sigma^2}{2}\right] dx$$

$$= \left\{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left[x-(\overline{X}+ju\sigma^2)\right]^2}{2\sigma^2}\right] dx\right\} \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right) = \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)$$

because $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left[x-(\overline{X}+ju\sigma^2)\right]^2}{2\sigma^2}\right] dx = 1$

Use Cauchy theorem for contour integral – net result is that it is just the same as integrating along the real axis.

Moments of Gaussian Random Variable

$$\frac{d^n}{du^n}\Phi(u)\Big|_{u=0} = j^n E[X^n] = j^n \overline{X^n}$$

$$\Phi_X(u) = \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)$$

$$\frac{d}{du}\Phi(u)\Big|_{u=0} = (j\overline{X} - u\sigma^2)\exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)\Big|_{u=0} = j\overline{X}$$

$$\frac{d^2}{du^2}\Phi(u)\Big|_{u=0} = -\sigma^2 \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right) + (j\overline{X} - u\sigma^2)^2 \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)\Big|_{u=0}$$

$$= -\sigma^2 - \overline{X}^2 = -\overline{X^2} = j^2\overline{X^2}$$

$$\overline{X^2} = \sigma^2 + \overline{X}^2$$

$$\frac{d^3}{du^3}\Phi(u)\Big|_{u=0} = \left(-3\sigma^2(j\overline{X} - u\sigma^2) + (j\overline{X} - u\sigma^2)^3\right)\exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)\Big|_{u=0}$$

$$= -3\sigma^2 j\overline{X} - j\overline{X}^3 = -j(3\sigma^2\overline{X} + \overline{X}^3) = j^3\overline{X^3}$$

$$\overline{X^3} = 3\sigma^2\overline{X} + \overline{X}^3$$

$$\frac{d^4}{du^4}\Phi(u)\Big|_{u=0} = \left(3\sigma^4 - 6\sigma^2(j\overline{X} - u\sigma^2)^2 + (j\overline{X} - u\sigma^2)^4\right)\exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)\Big|_{u=0}$$

$$= 3\sigma^4 + 6\sigma^2\overline{X}^2 + \overline{X}^4 = j^4\overline{X^4} = \overline{X^4}$$

Jointly Gaussian Random Variables

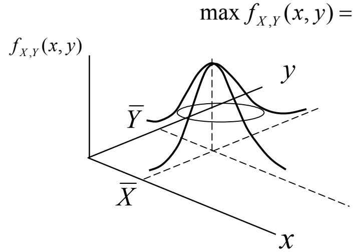
Two random variables are jointly Gaussian if their joint density function is of the form (sometimes called bivariate Gaussian)

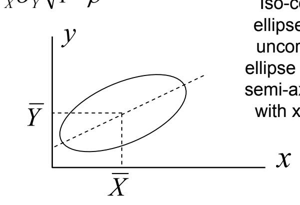
$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\overline{X})^2}{\sigma_X^2} - \frac{2\rho(x-\overline{X})(y-\overline{Y})}{\sigma_X\sigma_Y} + \frac{(y-\overline{Y})^2}{\sigma_Y^2}\right]\right\}$$

$$\overline{X} = E[X],\ \sigma_X^2 = E[(X-\overline{X})^2]$$
$$\overline{Y} = E[Y],\ \sigma_Y^2 = E[(Y-\overline{Y})^2]$$

$$\rho = \frac{E[(X-\overline{X})(Y-\overline{Y})]}{\sigma_X\sigma_Y}$$

$$\max f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$ at $(\overline{X}, \overline{Y})$





Iso-contour is an ellipse; if they are uncorrelated, the ellipse is not tilted – semi-axis in parallel with x and y axis.

Jointly Gaussian Random Variables

If the two Gaussian random variables are uncorrelated, $\rho = 0$,

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y}\exp\left\{-\frac{(x-\overline{X})^2}{2\sigma_X^2} - \frac{(y-\overline{Y})^2}{2\sigma_Y^2}\right\} = f_X(x)f_Y(y)$$

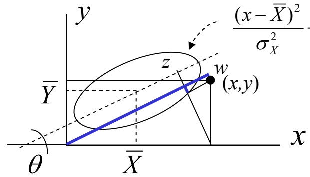
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left[\frac{-(x-\overline{X})^2}{2\sigma_X^2}\right], \quad -\infty < x < \infty$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y}\exp\left[\frac{-(y-\overline{Y})^2}{2\sigma_Y^2}\right], \quad -\infty < y < \infty$$

Uncorrelated Gaussian random variables are also statistically independent. Other properties of gaussian r.v.s include:

- Gaussian r.v.s are completely defined through their 1st- and 2nd-order moments, i.e., their means, variances, and covariances.
- Random variables produced by a linear transformation of jointly Gaussian r.v.s are also Gaussian.
- The conditional density functions defined over jointly Gaussian r.v.s is also Gaussian.

Linear Transformation of R.V.s



$$\frac{(x-\overline{X})^2}{\sigma_X^2} - \frac{2\rho(x-\overline{X})(y-\overline{Y})}{\sigma_X\sigma_Y} + \frac{(y-\overline{Y})^2}{\sigma_Y^2} = A = \text{constant}$$

Assume $\overline{X} = \overline{Y} = 0$,

$\rho = \pm 1,\ \frac{x^2}{\sigma_X^2} \pm \frac{2xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2} = K^2 \Rightarrow \frac{x}{\sigma_X} \pm \frac{y}{\sigma_Y} = \pm K$ two lines (but pdf is degenerate case)

$\rho = 0,\ \frac{x^2}{\sigma_X^2} + \frac{y^2}{\sigma_Y^2} = K^2 \Rightarrow$ an ellipse with semi-axis parallel to x and y axis

Covariance $C_{XY} = \rho\sigma_X\sigma_Y$

Let $W = X\cos\theta + Y\sin\theta \qquad \overline{W} = \overline{X}\cos\theta + \overline{Y}\sin\theta$

$\quad Z = -X\sin\theta + Y\cos\theta \qquad \overline{Z} = -\overline{X}\sin\theta + \overline{Y}\cos\theta$

$$C_{WZ} = E[(W-\overline{W})(Z-\overline{Z})] = E\{[(X-\overline{X})\cos\theta + (Y-\overline{Y})\sin\theta][-(X-\overline{X})\sin\theta + (Y-\overline{Y})\cos\theta]\}$$
$$= (\sigma_Y^2 - \sigma_X^2)\sin\theta\cos\theta + C_{XY}[\cos^2\theta - \sin^2\theta] = \tfrac{1}{2}(\sigma_Y^2 - \sigma_X^2)\sin 2\theta + C_{XY}\cos 2\theta$$

For W and Z to be uncorrelated, $C_{WZ} = 0$, which occurs at

$$\theta = \frac{1}{2}\tan^{-1}\left[\frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2 - \sigma_Y^2}\right].$$

Multivariate Gaussian

N random variables are jointly Gaussian if their joint density function is of the form (sometimes called multivariate Gaussian)

$$f_{X_1,X_2,\cdots,X_N}(x_1,x_2,\cdots,x_N) = \frac{\left|\mathbf{C}_X^{-1}\right|^{1/2}}{(2\pi)^{N/2}}\exp\left\{-\frac{(\mathbf{x}-\overline{\mathbf{X}})^t\mathbf{C}_X^{-1}(\mathbf{x}-\overline{\mathbf{X}})}{2}\right\}$$

where $\mathbf{x}^t = [x_1\ x_2\ x_3 \cdots x_N]$,
t is transpose and

$$\mathbf{x}-\overline{\mathbf{X}} = \begin{bmatrix} x_1-\overline{X}_1 \\ x_2-\overline{X}_2 \\ \vdots \\ x_N-\overline{X}_N \end{bmatrix} \qquad \mathbf{C}_X = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N1} & \cdots & C_{NN} \end{bmatrix}$$

$$C_{ij} = E[(X_i-\overline{X}_i)(X_j-\overline{X}_j)] = \begin{cases} \sigma_{X_i}^2, & i=j \\ C_{X_iX_j}, & i \neq j \end{cases}$$ is the covariance matrix

When $N = 2$,

$$\mathbf{C}_X = \begin{bmatrix} \sigma_{X_1}^2 & \rho\sigma_{X_1}\sigma_{X_2} \\ \rho\sigma_{X_1}\sigma_{X_2} & \sigma_{X_2}^2 \end{bmatrix} \qquad \mathbf{C}_X^{-1} = \frac{1}{(1-\rho^2)}\begin{bmatrix} 1/\sigma_{X_1}^2 & -\rho/\sigma_{X_1}\sigma_{X_2} \\ -\rho/\sigma_{X_1}\sigma_{X_2} & 1/\sigma_{X_2}^2 \end{bmatrix}$$

$$\left|\mathbf{C}_X^{-1}\right| = [\sigma_{X_1}^2\sigma_{X_2}^2(1-\rho^2)]^{-1}$$

Algorithmic Generation of Multiple R.V.s

- Generation of two gaussian r.v.s from two uniform r.v.s.
- Transformation of two gaussian r.v.s to achieve desired correlation.

X_1 and X_2 are two independent uniformly distributed r.v.s on $(0,1)$.

Let $Y_1 = \sqrt{-2\ln X_1}\cos(2\pi X_2)$ and $Y_2 = \sqrt{-2\ln X_1}\sin(2\pi X_2)$

Then, $X_1 = \exp\left(-\frac{Y_1^2+Y_1^2}{2}\right)$ and $X_2 = \frac{1}{2\pi}\tan^{-1}\frac{Y_2}{Y_1}$

We can show that $f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y_1^2}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y_2^2}{2}\right)$

That is, we have two independent 0-mean, 1-variance gaussian variables.
We use pseudo-random number generator to generate X_1 and X_2 first, then use the transformation to produce Y_1 and Y_2 .

Algorithmic Generation of Multiple R.V.s

Now, if we'd like the two Gaussian r.v.s to have arbitrary correlation and means, we need to perform linear transformation on the two gaussian r.v.s.. Recall

$$f_{Y_1,Y_2,\cdots,Y_N}(y_1,y_2,\cdots,y_N)=\frac{\left|\mathbf{C}_\mathbf{Y}^{-1}\right|^{1/2}}{(2\pi)^{N/2}}\exp\left\{-\frac{(\mathbf{y}-\overline{\mathbf{Y}})^t\mathbf{C}_\mathbf{Y}^{-1}(\mathbf{y}-\overline{\mathbf{Y}})}{2}\right\}$$

For the bivariate case:

$$\mathbf{C}=\begin{bmatrix}\sigma_{X_1}^2 & \rho\sigma_{X_1}\sigma_{X_2}\\ \rho\sigma_{X_1}\sigma_{X_2} & \sigma_{X_2}^2\end{bmatrix}\qquad \mathbf{C}^{-1}=\frac{1}{(1-\rho^2)}\begin{bmatrix}1/\sigma_{X_1}^2 & -\rho/\sigma_{X_1}\sigma_{X_2}\\ -\rho/\sigma_{X_1}\sigma_{X_2} & 1/\sigma_{X_2}^2\end{bmatrix}$$

$$\mathbf{y}=\mathbf{T}\mathbf{x},\quad \mathbf{T}^{-1}\mathbf{y}=\mathbf{x},\quad \mathbf{C}_\mathbf{X}=E[\mathbf{x}\mathbf{x}^t]=E\left[(\mathbf{T}^{-1}\mathbf{y})(\mathbf{T}^{-1}\mathbf{y})^t\right]=\mathbf{T}^{-1}E[\mathbf{y}\mathbf{y}^t]\mathbf{T}^{t^{-1}}=\mathbf{T}^{-1}\mathbf{C}_\mathbf{Y}\mathbf{T}^{t^{-1}}$$

Thus $\mathbf{C}_\mathbf{Y}=\mathbf{T}\mathbf{C}_\mathbf{X}\mathbf{T}^t=\mathbf{T}\mathbf{T}^t$, because $\mathbf{C}_\mathbf{X}=I$.

$$\mathbf{C}_\mathbf{Y}=\begin{bmatrix}\sigma_{Y_1}^2 & \rho\sigma_{Y_1}\sigma_{Y_2}\\ \rho\sigma_{Y_1}\sigma_{Y_2} & \sigma_{Y_2}^2\end{bmatrix}=\begin{bmatrix}\sigma_{Y_1} & 0\\ \rho\sigma_{Y_2} & \sigma_{Y_2}\sqrt{1-\rho^2}\end{bmatrix}\begin{bmatrix}\sigma_{Y_1} & \rho\sigma_{Y_2}\\ 0 & \sigma_{Y_2}\sqrt{1-\rho^2}\end{bmatrix}=\mathbf{T}\mathbf{T}^t,\quad \mathbf{T}=\begin{bmatrix}\sigma_{Y_1} & 0\\ \rho\sigma_{Y_2} & \sigma_{Y_2}\sqrt{1-\rho^2}\end{bmatrix}$$

Therefore, once two 0-mean, 1-var independent gaussian r.v.s are generated, we apply the transformation to obtain the desired results. If non-zero mean is required, just add the desired means to the transformed numbers.

Central Limit Theorem

- The probability distribution function of the sum of a large number of random variables approaches a Gaussian distribution. (We focus on statistically independent random variables here.)
- Let $\overline{X}$ and σ_X^2 be the means and variances, respectively, of N independent, identically distributed random variables $X_i, i=1,2,\cdots,N$; that is,

$$\overline{X}_1=\overline{X}_2=\cdots=\overline{X}_N=\overline{X}\ \text{ and }\ \sigma_{X_1}^2=\sigma_{X_2}^2=\cdots=\sigma_{X_N}^2=\sigma_X^2$$

- Consider the zero-mean, unit-variance r.v. associated with the sum $Y_N=X_1+X_2+\cdots+X_N$:

$$Z_N=(Y_N-\overline{Y}_N)\sigma_{Y_N}^{-1}=\left(\sum_{i=1}^N(X_i-\overline{X}_i)\right)\left(\sum_{i=1}^N\sigma_{X_i}^2\right)^{-1/2}=(\sqrt{N}\sigma_X)^{-1}\sum_{i=1}^N(X_i-\overline{X})$$

As $N\to\infty$, Z_N has a distribution that approaches Gaussian with zero-mean and unit-variance.

Central Limit Theorem – Proof *

Consider the characteristic function

$$\Phi_{Z_N}(\omega)=E[e^{j\omega Z_N}]=E\left\{\exp\left[j\omega\left(\sqrt{N}\sigma_X\right)^{-1}\sum_{i=1}^N(X_i-\overline{X})\right]\right\}$$

$$=\left\langle E\left\{\exp\left[j\omega\left(\sqrt{N}\sigma_X\right)^{-1}(X-\overline{X})\right]\right\}\right\rangle^N$$ where we use X to denote any of the N i.i.d. random variables.

But, $$\exp\left[\frac{j\omega(X-\overline{X})}{\sqrt{N}\sigma_X}\right]=1+\frac{j\omega(X-\overline{X})}{\sqrt{N}\sigma_X}+\left(\frac{j\omega}{\sqrt{N}\sigma_X}\right)^2\frac{(X-\overline{X})^2}{2}+\frac{R_N}{N}$$

$$E\left\{\exp\left[\frac{j\omega(X-\overline{X})}{\sqrt{N}\sigma_X}\right]\right\}=1-\frac{\omega^2}{2N}+\frac{E[R_N]}{N},$$ where $E[R_N]\to 0$, as $N\to\infty$

$$\ln[\Phi_{Z_N}(\omega)]=N\ln\left\{E\left[\exp\left(\frac{j\omega}{\sqrt{N}\sigma_X}(X-\overline{X})\right)\right]\right\}=N\ln\left(1-\frac{\omega^2}{2N}+\frac{E[R_N]}{N}\right)$$

Since $\ln(1-z)=-z-\frac{z^2}{2}-\frac{z^3}{3}+\cdots,|z|<1,\ \lim_{N\to\infty}\ln[\Phi_{Z_N}(\omega)]=-\omega^2/2.$

Therefore, $\lim_{N\to\infty}\Phi_{Z_N}(\omega)=e^{-\omega^2/2}$ and thus $\lim_{N\to\infty}Z_N$ is Gaussian with 0-mean, 1-var.

Central Limit Theorem – Another Proof *

$$Y_N=\frac{1}{N}\sum_{i=1}^N X_i,\ \overline{Y}_N=\frac{1}{N}\sum_{i=1}^N\overline{X}_i=\overline{X},$$

$$\sigma_{Y_N}^2=\frac{\sigma_X^2}{N}=\overline{Y_N^2}-\overline{Y}_N^2=\overline{Y_N^2}-\overline{X}^2=\frac{\overline{X^2}-\overline{X}^2}{N},\text{ and }\overline{X^2}=N\sigma_{Y_N}^2+\overline{X}^2.$$

$$\Phi_{Y_N}(\omega)=E[e^{j\omega Y_N}]=E\left\{\exp\left[\frac{j\omega}{N}\sum_{i=1}^N X_i\right]\right\}=\left\langle E\left\{\exp\left[\frac{j\omega X}{N}\right]\right\}\right\rangle^N$$

where we use X to denote any of the N i.i.d. random variables.

But, $$\exp\left[\frac{j\omega X}{N}\right]=1+\frac{j\omega}{N}\frac{X}{1}+\left(\frac{j\omega}{N}\right)^2\frac{X^2}{2}+R_N$$ due to Taylor series expansion

$$E\left\{\exp\left[\frac{j\omega X}{N}\right]\right\}=1+\frac{j\omega}{N}\overline{X}-\frac{\omega^2\overline{X^2}}{2N^2}+E[R_N]=1+\frac{j\omega\overline{X}}{N}-\frac{\omega^2(N\sigma_{Y_N}^2+\overline{X}^2)}{2N^2}+E[R_N]$$

$$=1+\frac{j\omega\overline{Y}_N}{N}-\frac{\omega^2\sigma_{Y_N}^2}{2N}-\frac{\omega^2\overline{X}^2}{2N^2}+E[R_N]$$ where $E[R_N]\to 0$, as $N\to\infty$.

Central Limit Theorem – Another Proof * (cont'd)

$$E\left\{\exp\left[\frac{j\omega X}{N}\right]\right\} = 1 + \frac{j\omega}{N}\overline{X} - \frac{\omega^2\overline{X^2}}{2N^2} + E[R_N] = 1 + \frac{j\omega\overline{X}}{N} - \frac{\omega^2(N\sigma_{Y_N}^2 + \overline{X}^2)}{2N^2} + E[R_N]$$

$$= 1 + \frac{j\omega\overline{Y}_N}{N} - \frac{\omega^2\sigma_{Y_N}^2}{2N} - \frac{\omega^2\overline{X}^2}{2N^2} + E[R_N]$$ where $E[R_N] \to 0$, as $N \to \infty$.

$$\ln[\Phi_{Y_N}(\omega)] = N\ln\left\{E\left[\exp\left(\frac{j\omega X}{N}\right)\right]\right\} = N\ln\left(1 + \frac{j\omega\overline{Y}_N}{N} - \frac{\omega^2\sigma_{Y_N}^2}{2N} - \frac{\omega^2\overline{X}^2}{2N^2} + E[R_N]\right)$$

Since $\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} + \cdots, |z| < 1,$

$$\lim_{N\to\infty}\ln[\Phi_{Y_N}(\omega)] = -N\left(-\frac{j\omega\overline{Y}_N}{N} + \frac{\omega^2\sigma_{Y_N}^2}{2N} + \frac{\omega^2\overline{X}^2}{2N^2}\right) = j\omega\overline{Y}_N - \frac{\omega^2\sigma_{Y_N}^2}{2}.$$

Therefore, $\lim_{N\to\infty}\Phi_{Y_N}(\omega) = e^{j\omega\overline{Y}_N - (\omega^2\sigma_{Y_N}^2/2)}$ which is the c.f. of a gaussian r.v. and thus Y_N is asymptotically Gaussian with mean $\overline{X}$, and variance σ_X^2 / N.

Central Limit Theorem - II

- Let $\overline{X}_i$ and $\sigma_{X_i}^2$ be the means and variances, respectively, of N independent random variables $X_i, i = 1, 2, \cdots, N$, which *may have arbitrary probability density functions.*
- The sum $Y_N = X_1 + X_2 + \cdots + X_N$, which has mean $\overline{Y}_N = \overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_N$ and variance $\sigma_{Y_N}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_N}^2$, has a probability distribution that asymptotically approaches Gaussian as $N \to \infty$.
- Sufficient conditions for the above to be true: $\sigma_{X_i}^2 > B > 0$ and $E\left[|X_i - \overline{X}_i|^3\right] < C, \quad i = 1, 2, \cdots, N,$ where B and C are some positive number; that is, no one dominant random variable in the sum
- Strictly speaking, the theorem only guarantees the distribution, not the density, to be asymptotically gaussian.

