ECE 3075A
Random Signals

Lecture 12

Sample Variance, Confidence Interval, Hypothesis Testing and Regression

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Sample Variance

- Do not confuse variance of sample mean with sample variance.
- Sample mean is an estimate of the mean of the population.
- A good estimate of the population mean does not readily represent good knowledge of the statistical properties of the population. Need to know some higher order statistics.
- What would be a reasonable estimate of the **variance of the population**?

With $X_i, i = 1, 2, \cdots, n$ being the r.v.s in the sample, the sample variance is defined as

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\bar{X}})^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n}X_j\right)^2$$

which is the average of the square of the difference between each r.v. and the sample mean.

Expectation of Sample Variance

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\bar{X}})^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n}X_j\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(X_i^2 - \frac{2X_i}{n}\sum_{j=1}^{n}X_j + \frac{1}{n^2}\sum_{j=1}^{n}X_j\sum_{k=1}^{n}X_k\right)$$

$$E[S^2] = \left(\frac{1}{n}\sum_{i=1}^{n}E[X_i^2]\right) - \left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}E[X_iX_j]\right)$$

$$= \overline{X^2} - \frac{1}{n^2}\left[(n^2-n)\overline{X}^2 + n\overline{X^2}\right] = \left(\frac{n-1}{n}\right)\overline{X^2} - \left(\frac{n-1}{n}\right)\overline{X}^2 = \frac{n-1}{n}\sigma^2$$

The expected value of the sample variance is not the true variance, and thus a biased estimate. We make the following adjustment

$$\tilde{S}^2 = \frac{n}{n-1}S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\bar{X}})^2, \quad \therefore\ E[\tilde{S}^2] = \frac{n}{n-1}E[S^2] = \sigma^2$$

Distributions of Parameter Estimates

- Without knowing the true distribution, we may still want to have some idea about the distribution of the parameter estimates.
- If $X_i, i = 1, 2, \cdots, n$ are Gaussian and independent with mean $\overline{X}$ and variance σ^2, then the sample mean is also Gaussian and the normalized random variable
$$Z = \frac{\hat{\bar{X}} - \overline{X}}{\sigma/\sqrt{n}}$$
is Gaussian with zero mean and unit variance. This true regardless of the size of the sample.
- If $X_i, i = 1, 2, \cdots, n$ are not Gaussian, then Z is asymptotically Gaussian as $n \to \infty$ according to the central limit theorem. As a rule of thumb, the asymptotic result is reached when $n \geq 30$.
- When n is not large enough to ensure normality, then the following normalized random variable Z has a Student's t distribution:
$$Z = \frac{\hat{\bar{X}} - \overline{X}}{\tilde{S}/\sqrt{n}} = \frac{\hat{\bar{X}} - \overline{X}}{S/\sqrt{n-1}}$$

Confidence Interval

- Sample mean is a point estimate of the population mean – a single value that we offer as an estimate of the true parameter.
- An alternative is to offer an interval estimate – e.g., the population mean is estimated to be within a certain interval with a certain probability.
- A q-percent confidence interval is the interval within which the parameter being estimated would lie with a probability of q/100. The value q is called the confidence level and calculated as:

$$\Pr\{\overline{X}-\frac{k\sigma}{\sqrt{n}}\le\hat{\overline{X}}\le\overline{X}+\frac{k\sigma}{\sqrt{n}}\}=q=100\int_{\overline{X}-k\sigma}^{\overline{X}+k\sigma}f_{\hat{\overline{X}}}(x)\,dx$$

- $\bar{x}-\frac{k\sigma}{\sqrt{n}}$ and $\bar{x}+\frac{k\sigma}{\sqrt{n}}$ are called the confidence limits; $\bar{x}$ is a "realized" sample average.

This is interpreted as: $\overline{X}$ (which is unknown and being estimated) will lie within the interval $\left(\bar{x}-\frac{k\sigma}{\sqrt{n}},\bar{x}+\frac{k\sigma}{\sqrt{n}}\right)$ at q% of the times sampling is made.

More on Confidence Interval

$$\Pr\{\overline{X}-\frac{k\sigma}{\sqrt{n}}\le\hat{\overline{X}}\le\overline{X}+\frac{k\sigma}{\sqrt{n}}\}=q=100\int_{\overline{X}-k\sigma}^{\overline{X}+k\sigma}f_{\hat{\overline{X}}}(x)\,dx$$

$\Rightarrow$ This is based on the previous result that for large n

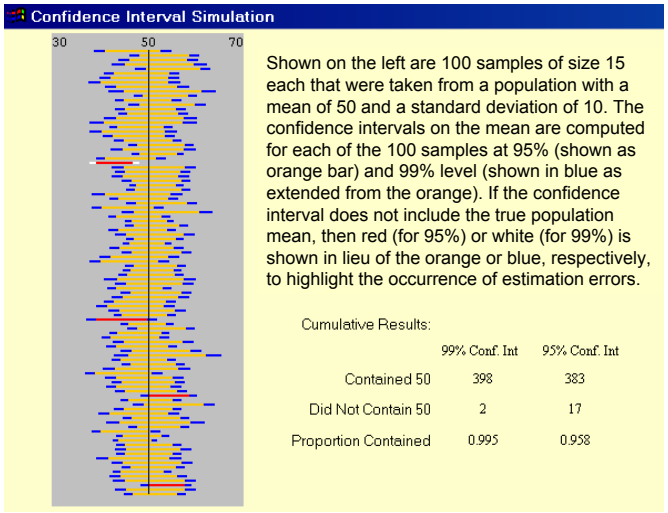
$$\hat{\overline{X}}=\frac{1}{n}\sum_{i=1}^{n}X_i,\quad E\left[\hat{\overline{X}}\right]=\overline{X},\quad \mathrm{var}\left(\hat{\overline{X}}\right)=\frac{\sigma_X^2}{n}$$

But after a sample is chosen, the realized sample mean $\bar{x}=\sum_{i=1}^{n}x_i$ and the sample variance $\tilde{s}^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$ are already calculated.

The interval for the purpose of predicting $\overline{X}$, $\overline{X}-\frac{k\sigma}{\sqrt{n}}\le\hat{\overline{X}}\le\overline{X}+\frac{k\sigma}{\sqrt{n}}$, is equivalent to $\hat{\overline{X}}-\frac{k\sigma}{\sqrt{n}}\le\overline{X}\le\hat{\overline{X}}+\frac{k\sigma}{\sqrt{n}}$ or in one realization $\bar{x}-\frac{ks}{\sqrt{n}}\le\overline{X}\le\bar{x}+\frac{ks}{\sqrt{n}}$ where the estimated mean and variance are used.

We thus say that $\overline{X}$ (the unknown being estimated) will lie within the interval $\left(\bar{x}-\frac{ks}{\sqrt{n}},\bar{x}+\frac{ks}{\sqrt{n}}\right)$ at q% confidence level.

Confidence Interval - Example



From Rice University

Hypothesis Testing

- A hypothesis is an assertion or a theory that one makes about a random observation, e.g., the lifetime of a component, the mean of a population, the presence of a signal together with a random noise, the distribution of a random variable.
- Hypothesis testing is a procedure one follows to accept or to reject said assertion in the context of statistical evidence.

Null Hypothesis – the theory

For example, in a clinical trial of a new drug, the null hypothesis might be that the new drug is no better, on average, than another (say currently used) drug.

H_0: there is no difference between the two drugs on average.

Alternative Hypothesis – opposite of the theory

For example, in a clinical trial of a new drug, the null hypothesis might be that the new drug is no better, on average, than another (say currently used) drug.

H_1: there is difference between the two drugs on average.

Likelihood Ratio Test

Gather evidence ξ and evaluate the likelihood of the hypotheses.

Likelihood: $\Pr\{\xi \mid H_0\}$ and $\Pr\{\xi \mid H_1\}$

Likelihood Ratio Test:

Accept H_0 if $\log \dfrac{\Pr\{\xi \mid H_0\}}{\Pr\{\xi \mid H_1\}} > \tau$; otherwise, reject H_0.

	Decision	
	Reject H_0	Accept H_0
H_0 is true	Type I error (miss)	Correct decision
H_1 is true	Correct decision	Type II error (false alarm)

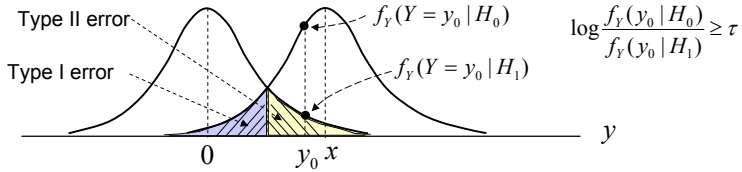
Simple Likelihood Ratio Test - Illustration

H_0: Signal x is present in the noisy observation y in which the additive noise v is independent gaussian with zero mean and variance σ_v^2

$$f_Y(y \mid H_0) = \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left\{-\frac{(y-x)^2}{2\sigma_v^2}\right\}$$

H_1: Signal x is not present in the noisy observation y with the same noise characteristics as in H_0

$$f_Y(y \mid H_1) = \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left\{-\frac{y^2}{2\sigma_v^2}\right\}$$



Curve Fitting

- Observations are made on two or more variables that may have a certain relationship – as in study of physics, for example. These data points, when plotted in a multi-dimensional space, form a scatter diagram.
- We use curve fitting to find a mathematical relationship, often simplified and parameterized, among the variables. The mathematical equation that relates the variables is called the regression equation which defines a regression curve, which fits the given observation data in some optimal sense, i.e., so-called criterion of goodness-of-fit.

Given n data points, in a 2-D example, $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ we choose to fit a curve defined by $y = g(x)$ to the data such that the fitting error defined as $D = \sum_{i=1}^{n} [y_i - g(x_i)]^2$ is minimized. This leads to a **least-square regression** curve. We can choose other criterion if we like.

Least-Square Regression

The function $y = g(x)$ is usually parameterized. For example, in linear regression $y = g(x) = a + bx$

Then, $D = \sum_{i=1}^{n} [y_i - g(x_i)]^2 = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$

$$\frac{\partial D}{\partial a} = 2\sum_{i=1}^{n} -(y_i - a - bx_i) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} (a + bx_i) = na + b\sum_{i=1}^{n} x_i$$

$$\frac{\partial D}{\partial b} = 2\sum_{i=1}^{n} -x_i(y_i - a - bx_i) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} (ax_i + bx_i^2) = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2$$

Let $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\overline{x^2} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\overline{xy} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$

$$\Longrightarrow \quad b = \frac{\overline{xy} - \bar{x}\,\bar{y}}{\overline{x^2} - (\bar{x})^2} \qquad a = \bar{y} - b\bar{x}$$

We can also generalize to m^{th} order polynomial $y = g(x) = \sum_{m=0}^{M} c_m x^m$.

But, beware of the problem with over-fitting.

Correlation between Data Sets

- It is often of interest to determine if two sets of data demonstrate observable statistical dependency or correlation – including if they come from the same source.
- The linear correlation coefficient (used as estimate of the true correlation coefficient) is obtained as:

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}} = \frac{\tilde{C}_{X,Y}}{\tilde{\sigma}_X \tilde{\sigma}_Y}$$ where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$\tilde{C}_{X,Y} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) \qquad \tilde{\sigma}_X = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad \tilde{\sigma}_Y = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

When n is large, r is approximately Gaussian. In general, the correlation is considered significant if $|r| > 0.5$.