ECE 3075A
Random Signals

Lecture 14
Random Processes

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Random Process and Outcomes of Experiment

- As in the random variable case which results from associating a random event with a set on the real line, a random process can be viewed as such a concept enlarged to include ***time***.
- We thus assign a time function to every outcome ξ
  $x(t,\xi)$
- The family of all such functions $X(t,\xi)$, with the corresponding outcomes forming a probability space, is called a random process. A short form notation is

$$x(t,\xi) \Rightarrow x(t)$$
$$X(t,\xi) \Rightarrow X(t)$$

Random Process

- A random process is a time function, the value of which at any instance of time is a random variable.
- Concepts and Notation:
  - Random Process: $X(t)$, an ensemble of functions
  - Sample function: $x(t)$

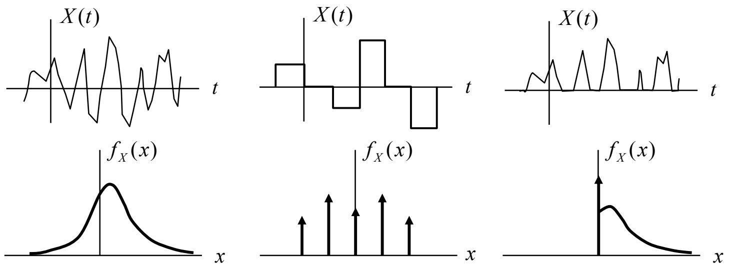
    A sample function is a realization of a random process, which implies an ensemble of functions that collectively form the observation space. A sample function is an element in the observation space.
  - At a specific time t_1 , $X(t_1) = X_1$ is a random variable.

Types of Processes

- Discrete ⇔ Continuous
- Deterministic ⇔ Non-deterministic
- Stationary ⇔ Non-stationary
- Ergodic ⇔ Non-ergodic

Continuous and Discrete Random Process

- If $X(t_i) = X_i$ for any t_i is a continuous random variable, $X(t)$ is a continuous random process.
- If $X(t_i) = X_i$ for any t_i is a discrete random variable, $X(t)$ is a discrete random process.
- If $X(t_i) = X_i$ has a mixed (continuous and discrete) distribution, $X(t)$ is a mixed random process.



Continuous   Discrete   Mixed

Example 5-2.2

A random time function has a mean value of 1 and an amplitude that has an exponential distribution. This function is multiplied by a sinusoid of unit amplitude and phase uniformly distributed over $(0, 2\pi)$

a) Classify the product as continuous, discrete, or mixed.
b) Classify the product after it has passed through an ideal hard limiter having an input-output characteristic given by

$$V_{out} = \mathrm{sgn}(V_{in})$$

c) Classify the product assuming the sinusoid is passed through a half-wave rectifier before multiplying the exponentially distributed time function and the sinusoid.

$$Y(t) = X(t)\sin\Theta$$

$$f_X(x) = \frac{1}{\eta}\exp\left(-\frac{x}{\eta}\right) \quad x \ge 0 \qquad \text{where } \eta = \overline{X} = 1$$
$$= 0 \quad x < 0$$

a) Since $Y(t)$ assumes continuous value, it is a continuous random process.
b) Since the sgn function assumes values of 1 and -1, it is a discrete process.
c) Since a half-wave rectifier would set the negative values of the sinusoid to zero, it creates a distinctive probability at value 0, and the output process is thus a mixed process.

Deterministic and Non-deterministic Processes

- A random process represents an ensemble of time functions, the value of which at any given time cannot be pre-determined or specified – thus a non-deterministic process.
- In contrast, a process is called deterministic if its value as a function of **time** can be pre-determined.

Example:

$X(t) = A\cos(\omega t + \Theta)$

where A and ω are constant and Θ is a random variable. Once the parameter Θ is determined, $X(t)$ as a function of time is entirely specified and is thus a deterministic process.

$X(t) = \sum_{n=0}^{\infty}\left[A_n\cos(2\pi nf_0 t) + B_n\sin(2\pi nf_0 t)\right]$ is also deterministic.

Example 5-3.1

A sample function of a random process, defined by

$$X(t) = A\exp(-\beta t) \quad t \ge 0$$

is observed to have the following values:

$X(1) = 1.21306$ and $X(2) = 0.73576.$

a) Find the value of A and β.
b) Find the value $X(3.2189)$

$$X(1) = A\exp(-\beta) = 1.21306$$
$$X(2) = A\exp(-2\beta) = 0.73576$$
$$X(1)/X(2) = \exp(\beta) = 1.21306/0.73576$$
$$\beta = \ln 1.21306 - \ln 0.73576 = 0.5$$
$$A = X(1)\exp(\beta) = (1.21306)^2/0.73576 = 2$$

Example 5-3.2

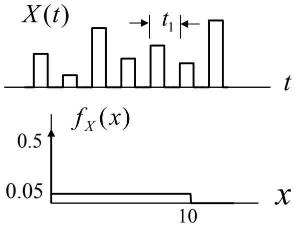
A random process has sample function of the form

$$X(t) = \sum_{n=-\infty}^{\infty} A_n f(t - nt_1)$$

where A_n are independent random variables that are uniformly distributed from 0 to 10 and

$$f(t) = 1 \quad 0 \le t \le t_1/2$$
$$= 0 \quad \text{elsewhere}$$

a) Is this process deterministic or non-deterministic? Why?
b) Is this process continuous, discrete, or mixed? Why?



a) Since the amplitude is not entirely specified as a function of time, it is non-deterministic.

b) Half of the time the process has value zero and half of the time uniformly distributed. It's mixed.

Stationary and Non-stationary Processes

- A random process is a time function whose value at any given time is a random variable. When a number, n, of time instances are considered, the corresponding r.v.s, $X(t_1), X(t_2), \cdots, X(t_n)$ have a joint density function $f_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n)$
- If all the marginals and joint density functions of the process do not depend on the choice of time origin, the process is said to be **stationary**. **Otherwise, non-stationary**.
- If a process satisfies:
  - The mean of any $X(t)$ does not depend on t, *i.e.*, $\overline{X}(t) = \overline{X}$
  - The correlation between any two r.v.s $E\{X(t_1)X(t_2)\}$ depends on the time difference $t_1 - t_2$ only

  It is called a **stationary process in the wide sense.**

More on Stationarity

- Stationarity in the wide sense is a special, but perhaps most useful, case of second-order stationarity.
- 2nd order stationarity requires

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

  for all t_1, t_2 and Δ. If we choose $\Delta = -t_1$, then it becomes

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; 0, t_2 - t_1) = f_X(x_1, x_2; 0, \tau)$$

- If then follows that the correlation function

$$R_{XX}(t_1, t_2) \equiv E[X(t_1)X(t_2)] = E[X(t_1)X(t_1 + \tau)] = R_{XX}(\tau)$$

  is a function of only the time difference and not the absolute time. 2nd order stationarity implies wide sense stationarity but the converse is not necessarily true.

Example

The following random process is wide sense stationary,

$$X(t) = A\cos(\omega_0 t + \Theta)$$

If it is assumed that A and ω_0 are constant and Θ is a uniformly distributed random variable on the interval $(0, 2\pi)$.

$$E[X(t)] = \int_0^{2\pi} \frac{1}{2\pi} A\cos(\omega_0 t + \theta)\, d\theta = 0$$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$
$$= E[A\cos(\omega_0 t + \Theta) A\cos(\omega_0 t + \omega_0 \tau + \Theta)]$$
$$= \frac{A^2}{2} E[\cos(\omega_0 \tau) + \cos(2\omega_0 t + \omega_0 \tau + 2\Theta)] = \frac{A^2}{2}\cos(\omega_0 \tau)$$

Thus the autocorrelation function depends only on τ and the mean is a constant. Therefore, $X(t)$ is wide-sense stationary.

Ergodic and Non-ergodic Processes

- Statistical properties such as marginal densities and joint densities of a **stationary** process remain the same regardless of the time origin.
- For example, $E[X(t)] = \overline{X}$
- A stationary process whose statistical properties (as determined by ensemble average) can be obtained by time average is called an ergodic process. In particular,

$$\overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x)\,dx = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X^n(t)\,dt$$

- If the above is not true, the process is non-ergodic. An ergodic process also implies that any sample function of the ensemble is a “typical” function that demonstrates the same statistical behavior as others in the ensemble.

More on Ergodicity

Let $\overline{x} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt \qquad \mathbf{R}_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)\,dt$

If $x(t)$ are sample functions of a random process, $\overline{x}$ and $\mathbf{R}_{XX}$ are random variables, with

$$E[\overline{x}] = \overline{X}$$
$$E[\mathbf{R}_{XX}(\tau)] = R_{XX}(\tau)$$

If the property of the random process is such that $\overline{x}$ and $\mathbf{R}_{XX}$ (and all other moments as well) as random variables have zero variances, then the process is an ergodic process. That is,

$$\overline{x} = \overline{X} \quad \text{and} \quad \mathbf{R}_{XX}(\tau) = R_{XX}(\tau)$$

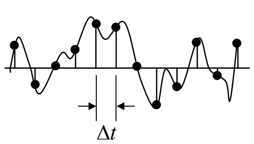
That also means, loosely put:
“Once you know one sample function from $-\infty$ to ∞, you know it all - everything about the statistical properties of that ergodic process.”

Ergodicity in Practical Setup

- Time average of an infinitely long function, particularly real world signals, can rarely be carried out analytically.
- In signal processing or computing, a time function is represented by a discrete time sequence (with arbitrary time origin):

$$X_1 = X(\Delta t),\ X_2 = X(2\Delta t),\ X_3 = X(3\Delta t), \cdots, X_N = X(N\Delta t)$$

- Then $\frac{1}{T}\int_0^T X(t)\,dt \quad \Rightarrow \quad \frac{1}{N}\sum_{i=1}^{N} X_i \equiv \hat{\overline{X}}$



$$E[\hat{\overline{X}}] = E\left[\frac{1}{N}\sum_{i=1}^{N} X_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[X_i] = \frac{1}{N}\sum_{i=1}^{N} \overline{X} = \overline{X}$$

$$E[(\hat{\overline{X}})^2] = E\left[\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j\right] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E[X_i X_j] \qquad \because\ E[X_i X_j] = \begin{cases} \overline{X^2}, & i = j \\ (\overline{X})^2, & i \neq j \end{cases}$$

$$= \frac{1}{N^2}[N\overline{X^2} + (N^2 - N)(\overline{X})^2] = \frac{1}{N}\sigma_X^2 + (\overline{X})^2$$

$$\text{var}(\hat{\overline{X}}) = E[(\hat{\overline{X}})^2] - (E[\hat{\overline{X}}])^2 = \sigma_X^2 / N \qquad \lim_{N\to\infty} \text{var}(\hat{\overline{X}}) = \lim_{N\to\infty} \sigma_X^2 / N = 0$$