

**ECE 3075A**  
***Random Signals***

**Lecture 14**  
**Random Processes**

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# Random Process and Outcomes of Experiment

- As in the random variable case which results from associating a random event with a set on the real line, a random process can be viewed as such a concept enlarged to include *time*.
- We thus assign a time function to every outcome  $\xi$   
 $x(t, \xi)$
- The family of all such functions  $X(t, \xi)$ , with the corresponding outcomes forming a probability space, is called a random process. A short form notation is

$$x(t, \xi) \Rightarrow x(t)$$

$$X(t, \xi) \Rightarrow X(t)$$

# Random Process

- A random process is a time function, the value of which at any instance of time is a random variable.
- Concepts and Notation:
  - Random Process:  $X(t)$ , an ensemble of functions
  - Sample function:  $x(t)$ 

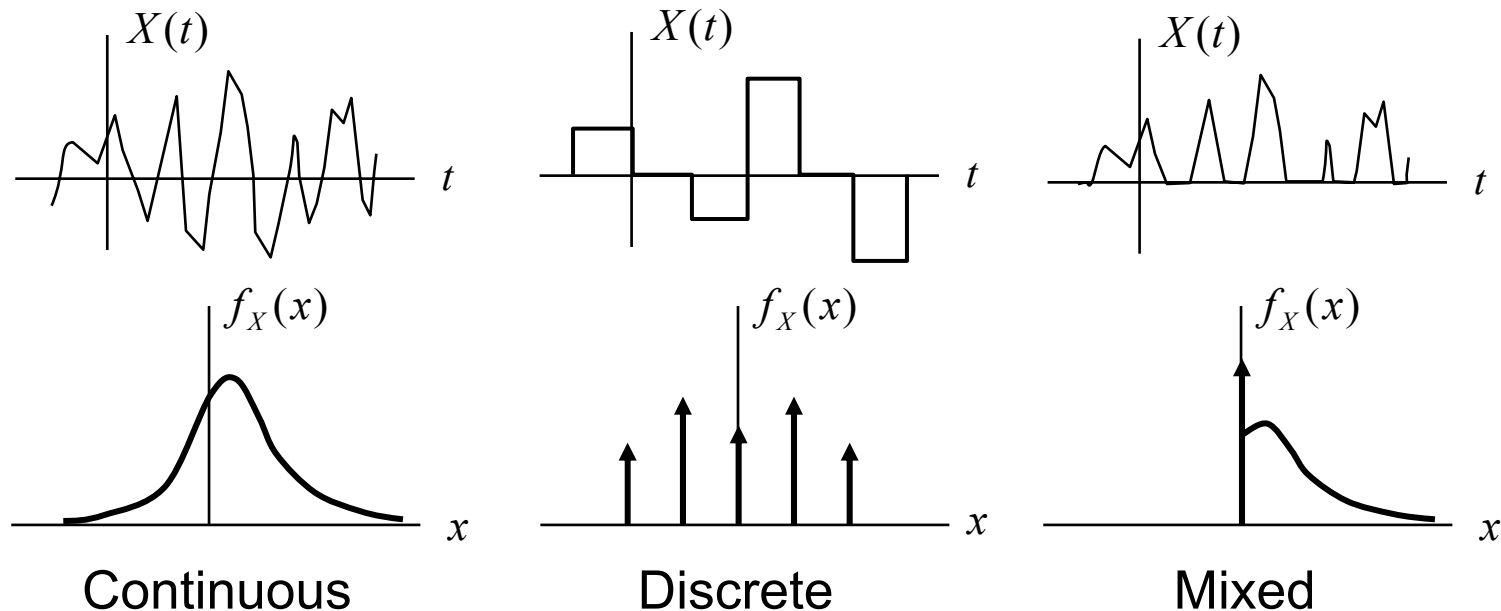
A sample function is a realization of a random process, which implies an ensemble of functions that collectively form the observation space. A sample function is an element in the observation space.
  - At a specific time  $t_1$ ,  $X(t_1) = X_1$  is a random variable.

# Types of Processes

- Discrete  $\Leftrightarrow$  Continuous
- Deterministic  $\Leftrightarrow$  Non-deterministic
- Stationary  $\Leftrightarrow$  Non-stationary
- Ergodic  $\Leftrightarrow$  Non-ergodic

# Continuous and Discrete Random Process

- If  $X(t_i) = X_i$  for any  $t_i$  is a continuous random variable,  $X(t)$  is a continuous random process.
- If  $X(t_i) = X_i$  for any  $t_i$  is a discrete random variable,  $X(t)$  is a discrete random process.
- If  $X(t_i) = X_i$  has a mixed (continuous and discrete) distribution,  $X(t)$  is a mixed random process.



## Example 5-2.2

A random time function has a mean value of 1 and an amplitude that has an exponential distribution. This function is multiplied by a sinusoid of unit amplitude and phase uniformly distributed over  $(0, 2\pi)$

- Classify the product as continuous, discrete, or mixed.
- Classify the product after it has passed through an ideal hard limiter having an input-output characteristic given by

$$V_{out} = \text{sgn}(V_{in})$$

- Classify the product assuming the sinusoid is passed through a half-wave rectifier before multiplying the exponentially distributed time function and the sinusoid.

$$Y(t) = X(t) \sin \Theta$$

$$f_x(x) = \begin{cases} \frac{1}{\eta} \exp\left(-\frac{x}{\eta}\right) & x \geq 0 \\ 0 & x < 0 \end{cases} \quad \text{where } \eta = \bar{X} = 1$$

- Since  $Y(t)$  assumes continuous value, it is a continuous random process.
- Since the sgn function assumes values of 1 and -1, it is a discrete process.
- Since a half-wave rectifier would set the negative values of the sinusoid to zero, it creates a distinctive probability at value 0, and the output process is thus a mixed process.

# Deterministic and Non-deterministic Processes

- A random process represents an ensemble of time functions, the value of which at any given time cannot be pre-determined or specified – thus a non-deterministic process.
- In contrast, a process is called deterministic if its value as a function of **time** can be pre-determined.

Example:

$$X(t) = A \cos(\omega t + \Theta)$$

where  $A$  and  $\omega$  are constant and  $\Theta$  is a random variable.

Once the parameter  $\Theta$  is determined,  $X(t)$  as a function of time is entirely specified and is thus a deterministic process.

$$X(t) = \sum_{n=0}^{\infty} [A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)] \quad \text{is also deterministic.}$$

## Example 5-3.1

A sample function of a random process, defined by

$$X(t) = A \exp(-\beta t) \quad t \geq 0$$

is observed to have the following values:

$$X(1) = 1.21306 \quad \text{and} \quad X(2) = 0.73576.$$

- a) Find the value of  $A$  and  $\beta$ .
- b) Find the value  $X(3.2189)$

$$X(1) = A \exp(-\beta) = 1.21306$$

$$X(2) = A \exp(-2\beta) = 0.73576$$

$$X(1) / X(2) = \exp(\beta) = 1.21306 / 0.73576$$

$$\beta = \ln 1.21306 - \ln 0.73576 = 0.5$$

$$A = X(1) \exp(\beta) = (1.21306)^2 / 0.73576 = 2$$



## Example 5-3.2

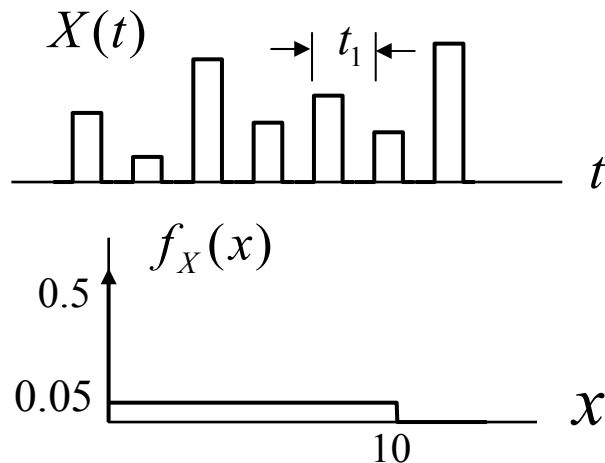
A random process has sample function of the form

$$X(t) = \sum_{n=-\infty}^{\infty} A_n f(t - nt_1)$$

where  $A_n$  are independent random variables that are uniformly distributed from 0 to 10 and

$$f(t) = \begin{cases} 1 & 0 \leq t \leq t_1/2 \\ 0 & \text{elsewhere} \end{cases}$$

- Is this process deterministic or non-deterministic? Why?
- Is this process continuous, discrete, or mixed? Why?



- Since the amplitude is not entirely specified as a function of time, it is non-deterministic.
- Half of the time the process has value zero and half of the time uniformly distributed. It's mixed.

# Stationary and Non-stationary Processes

- A random process is a time function whose value at any given time is a random variable. When a number,  $n$ , of time instances are considered, the corresponding r.v.s,  $X(t_1), X(t_2), \dots, X(t_n)$  have a joint density function

$$f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n)$$

- If all the marginals and joint density functions of the process do not depend on the choice of time origin, the process is said to be **stationary**. **Otherwise, non-stationary**.
- If a process satisfies:
  - The mean of any  $X(t)$  does not depend on  $t$ , *i.e.*,  $\bar{X}(t) = \bar{X}$
  - The correlation between any two r.v.s  $E\{X(t_1)X(t_2)\}$  depends on the time difference  $t_1 - t_2$  only

It is called a **stationary process in the wide sense**.

# More on Stationarity

- Stationarity in the wide sense is a special, but perhaps most useful, case of second-order stationarity.
- 2<sup>nd</sup> order stationarity requires

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

for all  $t_1, t_2$  and  $\Delta$ . If we choose  $\Delta = -t_1$ , then it becomes

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; 0, t_2 - t_1) = f_X(x_1, x_2; 0, \tau)$$

- It then follows that the correlation function

$$R_{XX}(t_1, t_2) \equiv E[X(t_1)X(t_2)] = E[X(t_1)X(t_1 + \tau)] = R_{XX}(\tau)$$

is a function of only the time difference and not the absolute time. 2<sup>nd</sup> order stationarity implies wide sense stationarity but the converse is not necessarily true.

# Example

The following random process is wide sense stationary,

$$X(t) = A \cos(\omega_0 t + \Theta)$$

If it is assumed that  $A$  and  $\omega_0$  are constant and  $\Theta$  is a uniformly distributed random variable on the interval  $(0, 2\pi)$ .

$$E[X(t)] = \int_0^{2\pi} \frac{1}{2\pi} A \cos(\omega_0 t + \theta) d\theta = 0$$

$$\begin{aligned} R_{XX}(t, t + \tau) &= E[X(t)X(t + \tau)] \\ &= E[A \cos(\omega_0 t + \Theta) A \cos(\omega_0 t + \omega_0 \tau + \Theta)] \\ &= \frac{A^2}{2} E[\cos(\omega_0 \tau) + \cos(2\omega_0 t + \omega_0 \tau + 2\Theta)] = \frac{A^2}{2} \cos(\omega_0 \tau) \end{aligned}$$

Thus the autocorrelation function depends only on  $\tau$  and the mean is a constant. Therefore,  $X(t)$  is wide-sense stationary.

# Ergodic and Non-ergodic Processes

- Statistical properties such as marginal densities and joint densities of a **stationary** process remain the same regardless of the time origin.
- For example,  $E[X(t)] = \bar{X}$
- A stationary process whose statistical properties (as determined by ensemble average) can be obtained by time average is called an ergodic process. In particular,

$$\overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x) dx = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X^n(t) dt$$

- If the above is not true, the process is non-ergodic. An ergodic process also implies that any sample function of the ensemble is a “typical” function that demonstrates the same statistical behavior as others in the ensemble.

## More on Ergodicity

$$\text{Let } \bar{x} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt \quad \mathbf{R}_{XX}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt$$

If  $x(t)$  are sample functions of a random process,  $\bar{x}$  and  $\mathbf{R}_{XX}$  are random variables, with

$$E[\bar{x}] = \bar{X}$$

$$E[\mathbf{R}_{XX}(\tau)] = R_{XX}(\tau)$$

If the property of the random process is such that  $\bar{x}$  and  $\mathbf{R}_{XX}$  (and all other moments as well) as random variables have zero variances, then the process is an ergodic process. That is,

$$\bar{x} = \bar{X} \quad \text{and} \quad \mathbf{R}_{XX}(\tau) = R_{XX}(\tau)$$

That also means, loosely put:

“Once you know one sample function from  $-\infty$  to  $\infty$ , you know it all - everything about the statistical properties of that ergodic process.”

# Ergodicity in Practical Setup

- Time average of an infinitely long function, particularly real world signals, can rarely be carried out analytically.
- In signal processing or computing, a time function is represented by a discrete time sequence (with arbitrary time origin):

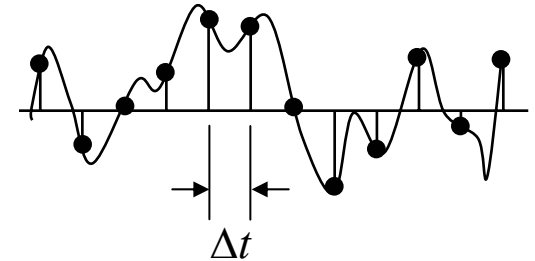
$$X_1 = X(\Delta t), X_2 = X(2\Delta t), X_3 = X(3\Delta t), \dots, X_N = X(N\Delta t)$$

- Then 
$$\frac{1}{T} \int_0^T X(t) dt \Rightarrow \frac{1}{N} \sum_{i=1}^N X_i \equiv \hat{X}$$

$$E[\hat{X}] = E\left[\frac{1}{N} \sum_{i=1}^N X_i\right] = \frac{1}{N} \sum_{i=1}^N E[X_i] = \frac{1}{N} \sum_{i=1}^N \bar{X} = \bar{X}$$

$$\begin{aligned} E[(\hat{X})^2] &= E\left[\frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N X_i X_j\right] = \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N E[X_i X_j] \\ &= \frac{1}{N^2} [N\bar{X}^2 + (N^2 - N)(\bar{X})^2] = \frac{1}{N} \sigma_X^2 + (\bar{X})^2 \end{aligned}$$

$$\text{var}(\hat{X}) = E[(\hat{X})^2] - (E[\hat{X}])^2 = \sigma_X^2 / N$$



$$\therefore E[X_i X_j] = \begin{cases} \bar{X}^2, & i = j \\ (\bar{X})^2, & i \neq j \end{cases}$$

$$\lim_{N \rightarrow \infty} \text{var}(\hat{X}) = \lim_{N \rightarrow \infty} \sigma_X^2 / N = 0$$