ECE 3075A

Random Signals

Lecture 15

Frequently Encountered Random Processes

School of Electrical and Computer Engineering
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Summer, 2003

Gaussian Random Processes

Let $X_1 = X(t_1), X_2 = X(t_2), \cdots, X_N = X(t_N)$

If $X_1, X_2, \cdots,$ and X_N are jointly Gaussian, the process is called a Gaussian random process.

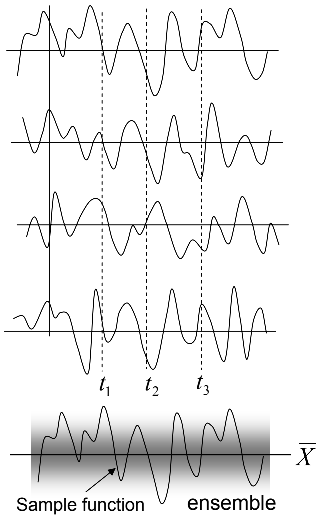
The joint density function is then

$$f_X(x_1, x_2, \cdots, x_N; t_1, t_2, \cdots, t_N)$$

$$= \frac{|\mathbf{C}_X^{-1}|}{(2\pi)^{N/2}} \exp\left\{-\frac{1}{2}[\mathbf{x}-\overline{\mathbf{X}}]^t \mathbf{C}_X^{-1}[\mathbf{x}-\overline{\mathbf{X}}]\right\}$$

$$\mathbf{x}-\overline{\mathbf{X}} = \begin{bmatrix} x_1 - \overline{X}_1 \\ x_2 - \overline{X}_2 \\ \vdots \\ x_N - \overline{X}_N \end{bmatrix} \quad \mathbf{C}_X = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N1} & \cdots & C_{NN} \end{bmatrix}$$

$$C_{ij} = E[(X_i - \overline{X}_i)(X_j - \overline{X}_j)] = \begin{cases} \sigma_{X_i}^2, & i = j \\ C_{X_i X_j}, & i \neq j \end{cases}$$



Gaussian Processes

Mean: $\overline{X}_i = E[X_i] = E[X(t_i)]$

Covariance Matrix:

$$\mathbf{C}_X = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N1} & \cdots & C_{NN} \end{bmatrix}$$

$$C_{ij} = C_{X_i X_j} = E[(X_i - \overline{X}_i)(X_j - \overline{X}_j)]$$
$$= E\{[X(t_i) - E[X(t_i)]][X(t_j) - E[X(t_j)]]\}$$
$$= C_{XX}(t_i, t_j)$$
$$= R_{XX}(t_i, t_j) - E[X(t_i)]E[X(t_j)]$$

A Gaussian random process is completely specified by the mean and the covariance matrix, or, equivalently, the autocorrelation function.

If the Gaussian process is wide sense stationary,

$\overline{X}_i = E[X(t_i)] = \overline{X}$ a constant w.r.t. time

$C_{XX}(t_i, t_j) = C_{XX}(t_j - t_i)$ and $R_{XX}(t_i, t_j) = R_{XX}(t_j - t_i)$ are functions of time difference only

If a Gaussian random process is wide sense stationary, it is also strict sense stationary.

Gaussian Process - Example

A wide sense stationary Gaussian process has a mean of $\overline{X} = 4$ and autocorrelation function $R_{XX}(\tau) = 25e^{-3|\tau|}$

Specify the joint density functions for three random variables $X(t_i), i = 1, 2, 3,$ at times $t_i = t_0 + [(i-1)/2]$ with t_0 a constant.

$t_j - t_i = (j-i)/2$ for i and $j = 1, 2, 3$

$$R_{XX}(\tau) = 25e^{-3|j-i|/2}$$

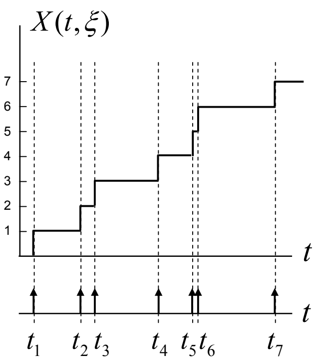
$$C_{XX}(t_j - t_i)$$
$$= R_{XX}(t_j - t_i) - E[X(t_i)]E[X(t_j)]$$
$$= 25e^{-3|j-i|/2} - 16$$

Practical insight: the autocorrelation of most if not all real signals will eventually go to zero (not necessarily monotonically though) as τ grows.

$$\mathbf{C}_{XX} = \begin{bmatrix} (25-16) & (25e^{-3/2}-16) & (25e^{-6/2}-16) \\ (25e^{-3/2}-16) & (25-16) & (25e^{-3/2}-16) \\ (25e^{-6/2}-16) & (25e^{-3/2}-16) & (25-16) \end{bmatrix}$$

Poisson Process

- Many events occur at random times. For example, the arrival of a customer at a bank, emission of an electron from the surface of a light-sensitive material, the failure of a certain component in a system.
- The accumulative number of such occurred events at any given time is a random process called Poisson or Poisson counting process.
- Events are assumed to occur:
  - Non-overlappingly, i.e., one event at a time even if they may be infinitesimally close;
  - Statistically independently



Time of occurrence of event

Other related terms:
- Point process
- Poisson arrival

Poisson Process

The probability of exactly k occurrences over a time interval $(0, t)$ is

$$\Pr\{X(t)=k\}=\frac{(\lambda t)^k}{k!}e^{-\lambda t}, \quad k=0,1,2,\cdots$$

where λ is the rate of occurrence, i.e., number of event occurrences per unit time and thus λt is the average number of occurrences in $(0,t)$

Example:

Assume car arrivals at a gas station occur at an average rate of 50 per hour. The station has only one pump and it takes 1 minute for each car to fuel. What is the probability that a waiting line will form at the pump?

A waiting line will form if two or more cars arrive in any 1-minute period. The probability of this occurring is one minus the probability that no or one car arrives in the duration, which is

With $\lambda t=\frac{50}{60}\times 1=\frac{5}{6}$,

$$\Pr\{\text{a waiting line forms}\}=1-\frac{(5/6)^0}{0!}e^{-5/6}-\frac{(5/6)^1}{1!}e^{-5/6}=0.2032$$

Poisson Density & Expectation

The probability density function of a Poisson process is thus

$$f_X(x)=\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t}\delta(x-k)$$

also a function of t; try to remember here the concept of a sample function

$$E[X(t)]=\int_{-\infty}^{\infty}xf_X(x)\,dx=\int_{-\infty}^{\infty}x\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t}\delta(x-k)\,dx$$

$$=\sum_{k=0}^{\infty}\frac{k(\lambda t)^k}{k!}e^{-\lambda t}=\lambda t\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t}=\lambda t$$

$$[0+\lambda t+2(\lambda t)^2+3(\lambda t)^3+\cdots]e^{-\lambda t}$$

$$=\lambda t[1+2(\lambda t)+3(\lambda t)^2+\cdots]e^{-\lambda t}=\lambda t\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

$$E[X^2(t)]=\int_{-\infty}^{\infty}x^2f_X(x)\,dx=\int_{-\infty}^{\infty}x^2\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t}\delta(x-k)\,dx$$

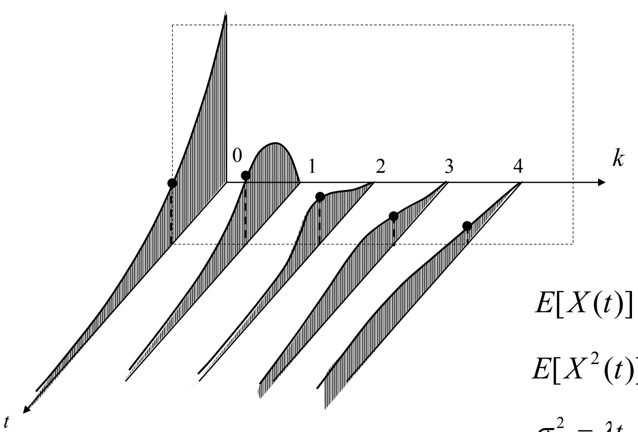
$$=\sum_{k=0}^{\infty}\frac{k^2(\lambda t)^k}{k!}e^{-\lambda t}=\lambda t(1+\lambda t)\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t}=\lambda t(1+\lambda t)$$

$$\sigma_X^2=E[X^2]-(E[X])^2=\lambda t(1+\lambda t)-(\lambda t)^2=\lambda t=E[X]$$

The variance equals the mean.

Can a Poisson process be a stationary process?

The Poisson Distribution



$$E[X(t)]=\lambda t$$

$$E[X^2(t)]=\lambda t(1+\lambda t)$$

$$\sigma_X^2=\lambda t$$

Poisson Distributions - Example

A parking lot attendant is also a statistician who found that there is a probability of 0.14 that 3 cars will arrive at the lot in a 5-minute interval. If indeed the car arrival follows Poisson process, what is the probability that no more than 2 cars arrive in a 10-minute period?

$$\Pr\{X(t)=k\}=\frac{(\lambda t)^k}{k!}e^{-\lambda t},\quad k=0,1,2,\cdots$$

$$\Pr\{X=3\}=\frac{(\lambda 5)^3}{3!}e^{-\lambda 5}=0.14 \quad\Rightarrow\quad 3\ln(\lambda 5)-\lambda 5=\ln(6\times 0.14)$$

$$3\ln\lambda=-3\ln 5+\lambda 5-0.174353=5(\lambda-1)\quad\Rightarrow\quad 0.6\ln\lambda=\lambda-1\quad\Rightarrow\quad \lambda=1$$

The average arrival rate is thus one car per minute.

$$\Pr\{X(10)=0\}=\frac{(1\times 10)^0}{0!}e^{-1\times 10}=0.0000454$$

$$\Pr\{X(10)=1\}=\frac{(1\times 10)^1}{1!}e^{-1\times 10}=0.000454$$

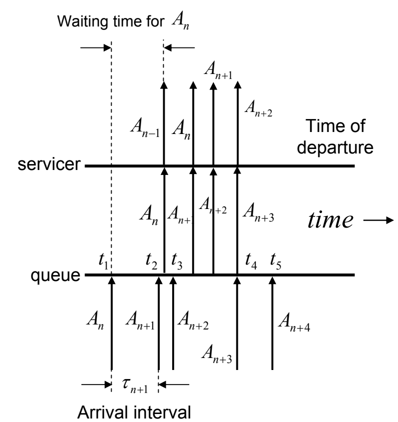
$$\Pr\{X(10)=2\}=\frac{(1\times 10)^2}{2!}e^{-1\times 10}=0.00277$$

Therefore,

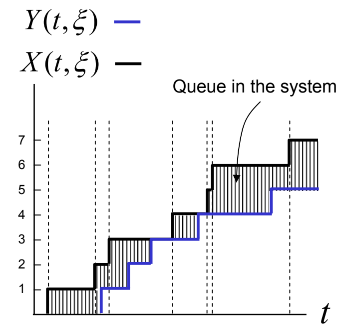
$$\Pr\{X(10)<3\}=0.003265$$

Note: at t=10, $E[X]$=10 cars, and the probability is 0.1251, much higher than the number 0.003265 above.

Use of Poisson Process in Queuing Analysis



Time of "customer" arrival



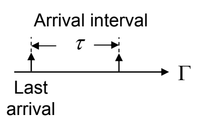
Both the arrival and the departure process, $X(t,\xi)$ and $Y(t,\xi)$, can be modeled as a Poisson process.

Arrival Interval of A Poisson Process

- Probability of arrival interval, τ, as a random variable is the same as the probability that there is no arrival during that interval.

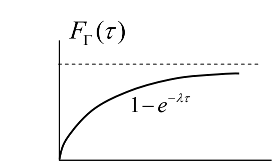
$$\Pr\{X(\tau)=0\}=\frac{(\lambda\tau)^0}{0!}e^{-\lambda\tau}=e^{-\lambda\tau}$$

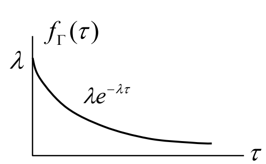
$$\begin{aligned}F_\Gamma(\tau)&=\Pr\{\text{arrival interval}\le\tau\}\\&=1-\Pr\{\text{no arrival in }(0,\tau)\}\\&=1-\Pr\{X(\tau)=0\}=1-e^{-\lambda\tau},\quad \tau\ge 0\end{aligned}$$



$$f_\Gamma(\tau)=\frac{d}{d\tau}F_\Gamma(\tau)=\lambda e^{-\lambda\tau},\quad \tau\ge 0$$

An exponential distribution!



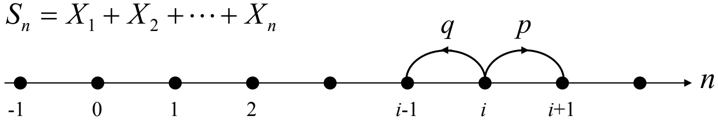


Random Walks

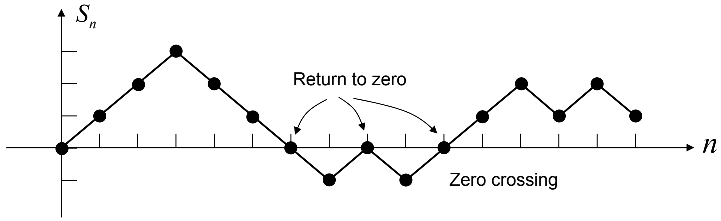
- $X_1, X_2, \cdots, X_n, \cdots$ are a sequence of independent random variables that assume values +1 and –1 with probability p and q=1-p, respectively. (The value of 1 is arbitrary subject to scaling.)
- Let $S_n=X_1+X_2+\cdots+X_n$ be the partial sum with $S_0=0$. The partial sum represents the accumulated positive or negative results at the end of nth trial.
- Many real life phenomena can be modeled by a random walk process.
  - The stock value variations of a particular stock;
  - The number of sign changes (or zero crossing, or level crossing) of a signal
  - The motion of gas molecules in a diffusion process
- If $p=q$, it is called a symmetric random walk; otherwise, an asymmetric random walk.

Random Walks

$S_n = X_1 + X_2 + \cdots + X_n$



Integer state representation: S_n is the state the system is in.



Time function representation: S_n is a function of "time" index.

Random Walks

Let the event $\{ S_n = r \}$ be the event "at time n the system is at point r," and $p_{n,r} = \Pr\{S_n = r\}$

$$p_{n,r} = \Pr\{S_n = r\} = \binom{n}{k} p^k q^{n-k}$$

where k is the number of +1's in n trials and n-k the number of –1's, and the net gain, which is r, is

$$r = k - (n-k) = 2k - n \quad \text{or} \quad k = (n+r)/2$$

Thus,

$$p_{n,r} = \binom{n}{(n+r)/2} p^{(n+r)/2} q^{(n-r)/2}$$

where $(n+r)/2$ is an integer between 0 and n inclusive; that is, n and r must be both odd or both even at the same time.

Random Walks - Examples

What is the probability that $S_n = r = n$?

$p_{n,n} = \Pr\{S_n = n\} = \Pr\{$walk in the same positive direction for n times$\}$

$$p_{n,n} = \Pr\{S_n = n\} = \binom{n}{n} p^k q^{n-k} = p^n$$

What is the probability that $S_n = r = 0$?

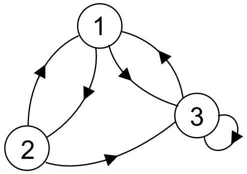
$$p_{n,0} = \binom{n}{n/2} p^{n/2} q^{n/2} = \binom{n}{n/2} p^{n/2} (1-p)^{n/2}$$

The polynomial coefficient indicates that there are $\binom{n}{n/2} = \frac{n!}{(n/2)!(n/2)!}$ paths that contain equal number of upward and downward steps.

What is the probability that a random walk process will reach a place within 2 steps of the peak, given that n steps have taken place?

Markov Chain and Processes

- Markov Process Taxonomy
  - By state space
    - Discrete: Markov Chain
    - Continuous: Markov process
  - By time of transition
    - Continuous time
    - Discrete time



Discrete state

At state 1, $X=b_1$

At state 2, $X=b_2$

At state 3, $X=b_3$

$X(t) = b_i, \quad i = 1, 2, \cdots, N$ depending on the state the system is in.

Markov Property:

$$\Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \cdots, X(t_1) = x_1\}$$
$$= \Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\} \quad \text{where } t_1 < t_2 < \cdots < t_n$$

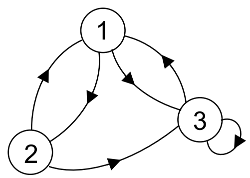
Finite State Markov Chain

- Markov Chain

$$\Pr\{X(t_n)=x_n, X(t_{n-1})=x_{n-1}, X(t_{n-2})=x_{n-2},\cdots,X(t_1)=x_1\}$$
$$=\Pr\{X(t_n)=x_n \mid X(t_{n-1})=x_{n-1}, X(t_{n-2})=x_{n-2},\cdots,X(t_1)=x_1\}$$
$$\bullet \Pr\{X(t_{n-1})=x_{n-1} \mid X(t_{n-2})=x_{n-2},\cdots,X(t_1)=x_1\}\bullet\cdots\bullet\Pr\{X(t_2)=x_2 \mid X(t_1)=x_1\}\bullet\Pr\{X(t_1)=x_1\}$$
$$=\Pr\{X(t_n)=x_n \mid X(t_{n-1})=x_{n-1}\}\Pr\{X(t_{n-1})=x_{n-1} \mid X(t_{n-2})=x_{n-2}\}\bullet\cdots\bullet\Pr\{X(t_2)=x_2 \mid X(t_1)=x_1\}\bullet\Pr\{X(t_1)=x_1\}$$

- An N-state (1st order) Markov chain is characterized by an initial state probability array, $\{\Pr[X(t_1)]\}_{\text{all states}}$, and a transition probability matrix $\mathbf{A}$,

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & \cdots & & \\ a_{N1} & a_{N2} & \cdots & a_{NN}\end{bmatrix} \qquad \sum_{j=1}^{N} a_{ij}=1$$



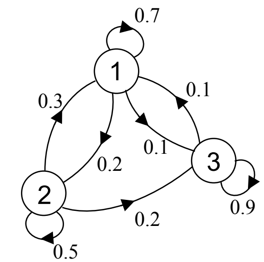
where $a_{ij}=\Pr\{X(t_n)=b_j \mid X(t_{n-1})=b_i\}$ where $t_{n+1}<t_n$.

Markov Chain - Example

- A finite-state Markov Chain is governed by the following parameters:

The initial probability vector: $[0.7 \quad 0.2 \quad 0.1]^t$

The state transition matrix: $\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.0 & 0.9 \end{bmatrix}$

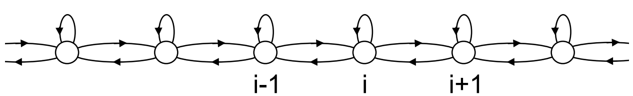


What is the probability that the system goes through the following sequence of states: (1, 3, 3, 1, 3, 1, 2, 1, 1, 3)

$$\Pr\{(1,3,3,1,3,1,2,1,1,3)\}$$
$$=0.7\times0.1\times0.9\times0.1\times0.1\times0.1\times0.2\times0.3\times0.7\times0.1=2.646\times10^{-7}$$

Other Random Processes

- Birth-Death Processes: special case of Markov Process in that the **state** can only move up (one birth) or down (one death) by one or stay the same (no birth or death); however, the value associated with a state can be arbitrary (unlike random walk in which the random variable changes **value** by the same amount, say 1, up or down).



- Renewal Processes: related to random walk but with interest in counting the transitions that takes place as a function of time.