

ECE 3075A
Random Signals

Lecture 15

Frequently Encountered Random Processes

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Gaussian Random Processes

Let $X_1 = X(t_1), X_2 = X(t_2), \dots, X_N = X(t_N)$

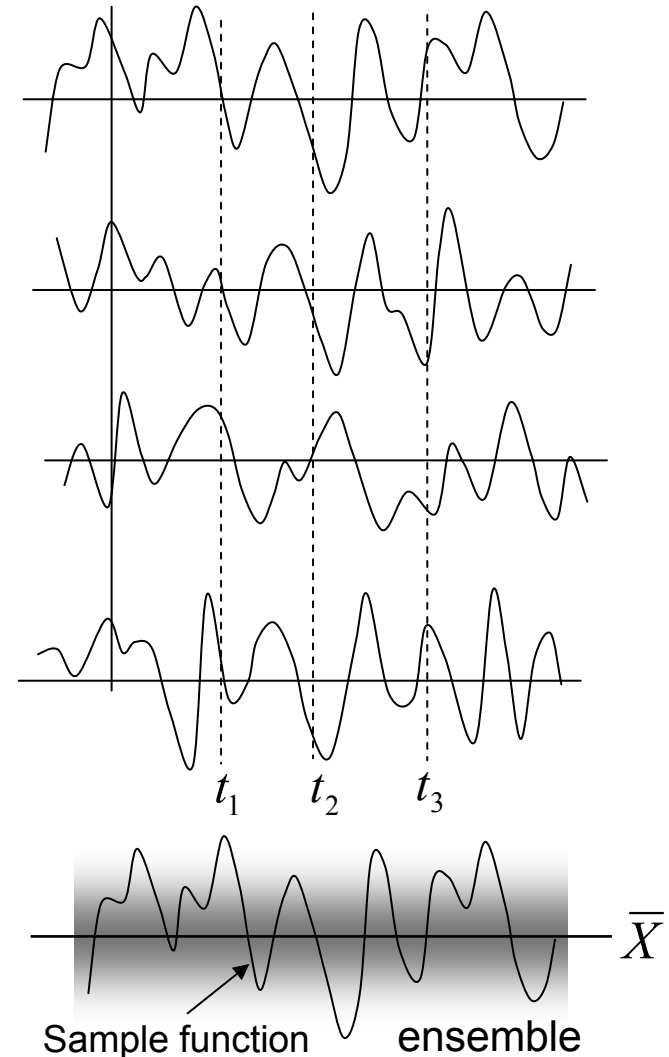
If $X_1, X_2, \dots,$ and X_N are jointly Gaussian, the process is called a Gaussian random process.

The joint density function is then

$$f_X(x_1, x_2, \dots, x_N; t_1, t_2, \dots, t_N) = \frac{|\mathbf{C}_X^{-1}|}{(2\pi)^{N/2}} \exp\left\{-\frac{1}{2}[\mathbf{x} - \bar{\mathbf{X}}]^t \mathbf{C}_X^{-1} [\mathbf{x} - \bar{\mathbf{X}}]\right\}$$

$$\mathbf{x} - \bar{\mathbf{X}} = \begin{bmatrix} x_1 - \bar{X}_1 \\ x_2 - \bar{X}_2 \\ \vdots \\ x_N - \bar{X}_N \end{bmatrix} \quad \mathbf{C}_X = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

$$C_{ij} = E[(X_i - \bar{X}_i)(X_j - \bar{X}_j)] = \begin{cases} \sigma_{X_i}^2, & i = j \\ C_{X_i X_j}, & i \neq j \end{cases}$$



Gaussian Processes

Mean: $\bar{X}_i = E[X_i] = E[X(t_i)]$

Covariance Matrix:

$$\mathbf{C}_X = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N1} & \cdots & C_{NN} \end{bmatrix} \quad \begin{aligned} C_{ij} &= C_{X_i X_j} = E[(X_i - \bar{X}_i)(X_j - \bar{X}_j)] \\ &= E\{[X(t_i) - E[X(t_i)]] [X(t_j) - E[X(t_j)]]\} \\ &= C_{XX}(t_i, t_j) \\ &= R_{XX}(t_i, t_j) - E[X(t_i)]E[X(t_j)] \end{aligned}$$

A Gaussian random process is completely specified by the mean and the covariance matrix, or, equivalently, the autocorrelation function.

If the Gaussian process is wide sense stationary,

$$\bar{X}_i = E[X(t_i)] = \bar{X} \quad \text{a constant w.r.t. time}$$

$C_{XX}(t_i, t_j) = C_{XX}(t_j - t_i)$ and $R_{XX}(t_i, t_j) = R_{XX}(t_j - t_i)$ are functions of time difference only

If a Gaussian random process is wide sense stationary, it is also strict sense stationary.

Gaussian Process - Example

A wide sense stationary Gaussian process has a mean of $\bar{X} = 4$ and autocorrelation function $R_{XX}(\tau) = 25e^{-3|\tau|}$

Specify the joint density functions for three random variables $X(t_i)$, $i = 1, 2, 3$, at times $t_i = t_0 + [(i-1)/2]$ with t_0 a constant.

$$t_j - t_i = (j-i)/2 \text{ for } i \text{ and } j = 1, 2, 3$$

$$R_{XX}(\tau) = 25e^{-3|j-i|/2}$$

$$C_{XX}(t_j - t_i)$$

$$= R_{XX}(t_j - t_i) - E[X(t_i)]E[X(t_j)]$$

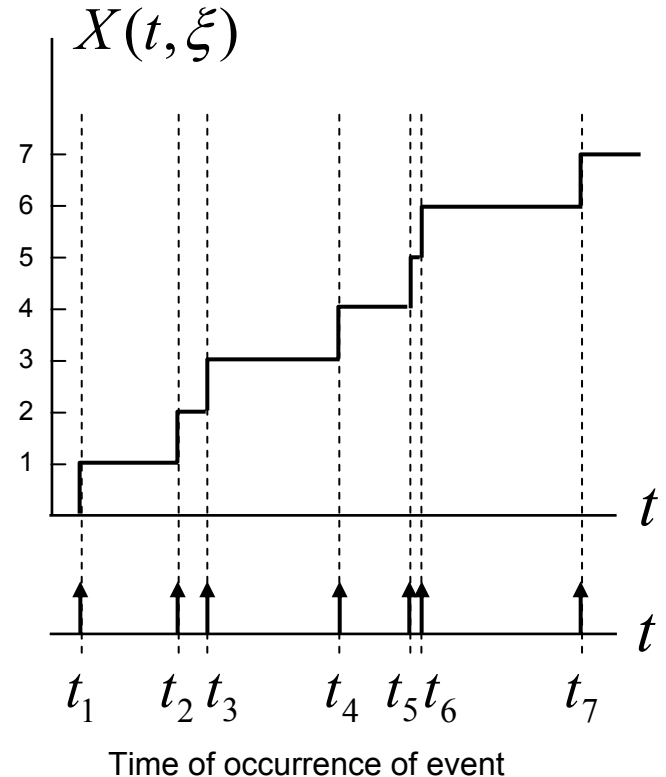
$$= 25e^{-3|j-i|/2} - 16$$

$$\mathbf{C}_{XX} = \begin{bmatrix} (25-16) & (25e^{-3/2} - 16) & (25e^{-6/2} - 16) \\ (25e^{-3/2} - 16) & (25-16) & (25e^{-3/2} - 16) \\ (25e^{-6/2} - 16) & (25e^{-3/2} - 16) & (25-16) \end{bmatrix}$$

Practical insight: the autocorrelation of most if not all real signals will eventually go to zero (not necessarily monotonically though) as τ grows.

Poisson Process

- Many events occur at random times. For example, the arrival of a customer at a bank, emission of an electron from the surface of a light-sensitive material, the failure of a certain component in a system.
- The accumulative number of such occurred events at any given time is a random process called Poisson or Poisson counting process.
- Events are assumed to occur:
 - Non-overlappingly, i.e., one event at a time even if they may be infinitesimally close;
 - Statistically independently



Other related terms:

- Point process
- Poisson arrival

Poisson Process

The probability of exactly k occurrences over a time interval $(0, t)$ is

$$\Pr\{X(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2, \dots$$

where λ is the rate of occurrence, i.e., number of event occurrences per unit time and thus λt is the average number of occurrences in $(0, t)$

Example:

Assume car arrivals at a gas station occur at an average rate of 50 per hour. The station has only one pump and it takes 1 minute for each car to fuel. What is the probability that a waiting line will form at the pump?

A waiting line will form if two or more cars arrive in any 1-minute period. The probability of this occurring is one minus the probability that no or one car arrives in the duration, which is

$$\text{With } \lambda t = \frac{50}{60} \times 1 = \frac{5}{6},$$

$$\Pr\{\text{a waiting line forms}\} = 1 - \frac{(5/6)^0}{0!} e^{-5/6} - \frac{(5/6)^1}{1!} e^{-5/6} = 0.2032$$

Poisson Density & Expectation

The probability density function of a Poisson process is thus

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \delta(x-k)$$

also a function of t ; try to remember here the concept of a sample function

$$E[X(t)] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \delta(x-k) dx$$

$$= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k}{k!} e^{-\lambda t} = \lambda t \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \lambda t$$

$[0 + \lambda t + 2(\lambda t)^2 + 3(\lambda t)^3 + \dots] e^{-\lambda t}$
 $= \lambda t [1 + 2(\lambda t) + 3(\lambda t)^2 + \dots] e^{-\lambda t} = \lambda t \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

$$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_{-\infty}^{\infty} x^2 \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \delta(x-k) dx$$

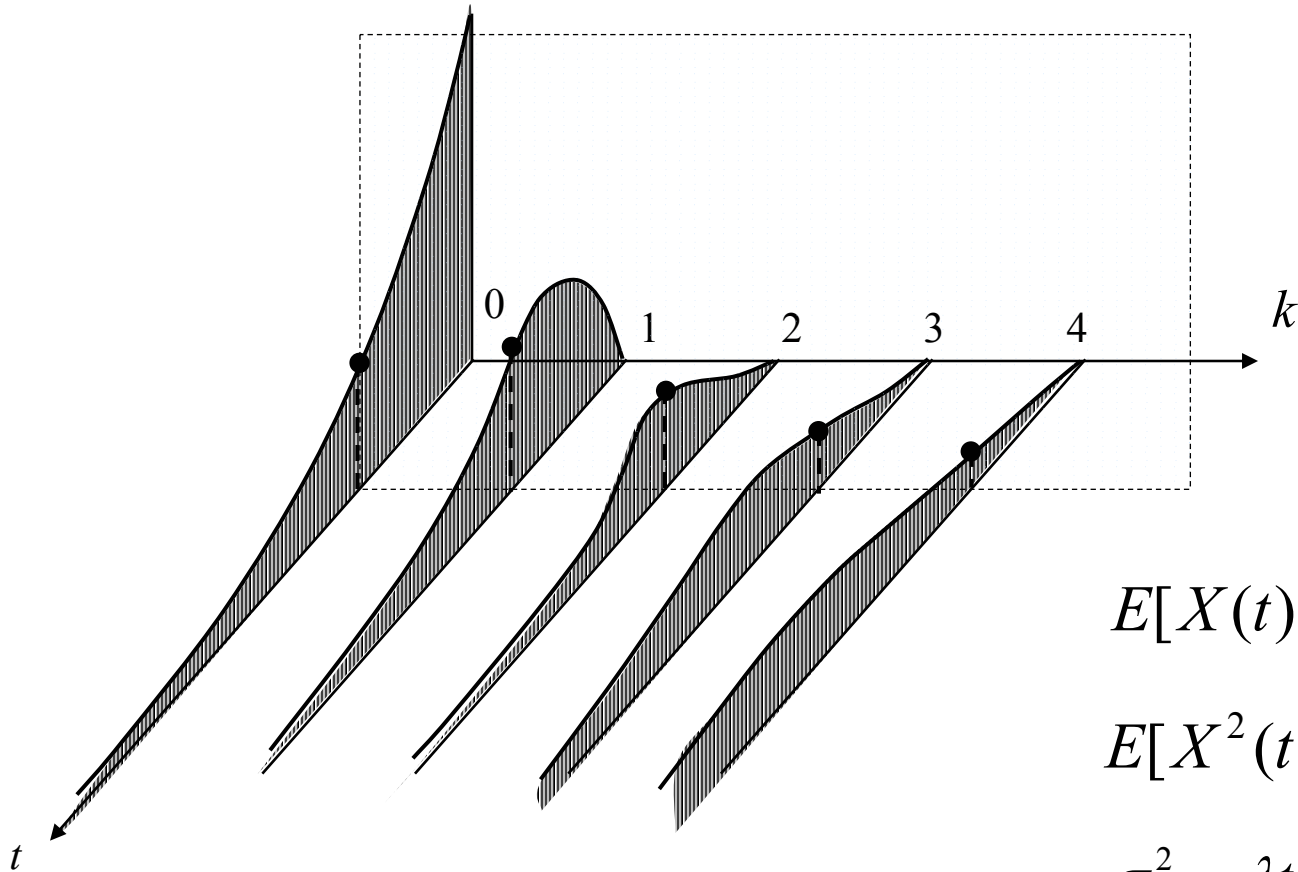
$$= \sum_{k=0}^{\infty} \frac{k^2 (\lambda t)^k}{k!} e^{-\lambda t} = \lambda t (1 + \lambda t) \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \lambda t (1 + \lambda t)$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \lambda t (1 + \lambda t) - (\lambda t)^2 = \lambda t = E[X]$$

The variance equals the mean.

Can a Poisson process be a stationary process?

The Poisson Distribution



$$E[X(t)] = \lambda t$$

$$E[X^2(t)] = \lambda t(1 + \lambda t)$$

$$\sigma_X^2 = \lambda t$$

Poisson Distributions - Example

A parking lot attendant is also a statistician who found that there is a probability of 0.14 that 3 cars will arrive at the lot in a 5-minute interval. If indeed the car arrival follows Poisson process, what is the probability that no more than 2 cars arrive in a 10-minute period?

$$\Pr\{X(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2, \dots$$

$$\Pr\{X = 3\} = \frac{(\lambda 5)^3}{3!} e^{-\lambda 5} = 0.14 \Rightarrow 3 \ln(\lambda 5) - \lambda 5 = \ln(6 \times 0.14)$$

$$3 \ln \lambda = -3 \ln 5 + \lambda 5 - 0.174353 = 5(\lambda - 1) \Rightarrow 0.6 \ln \lambda = \lambda - 1 \Rightarrow \lambda = 1$$

The average arrival rate is thus one car per minute.

$$\Pr\{X(10) = 0\} = \frac{(1 \times 10)^0}{0!} e^{-1 \times 10} = 0.0000454$$

$$\Pr\{X(10) = 1\} = \frac{(1 \times 10)^1}{1!} e^{-1 \times 10} = 0.000454$$

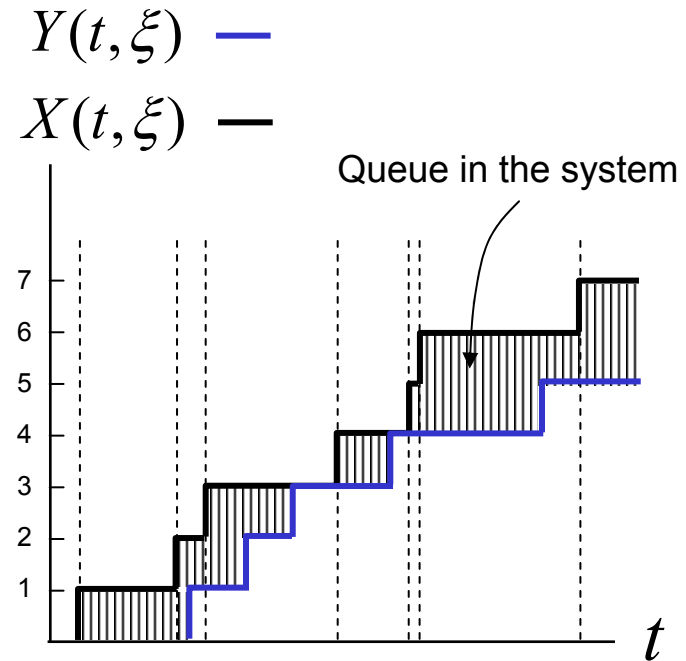
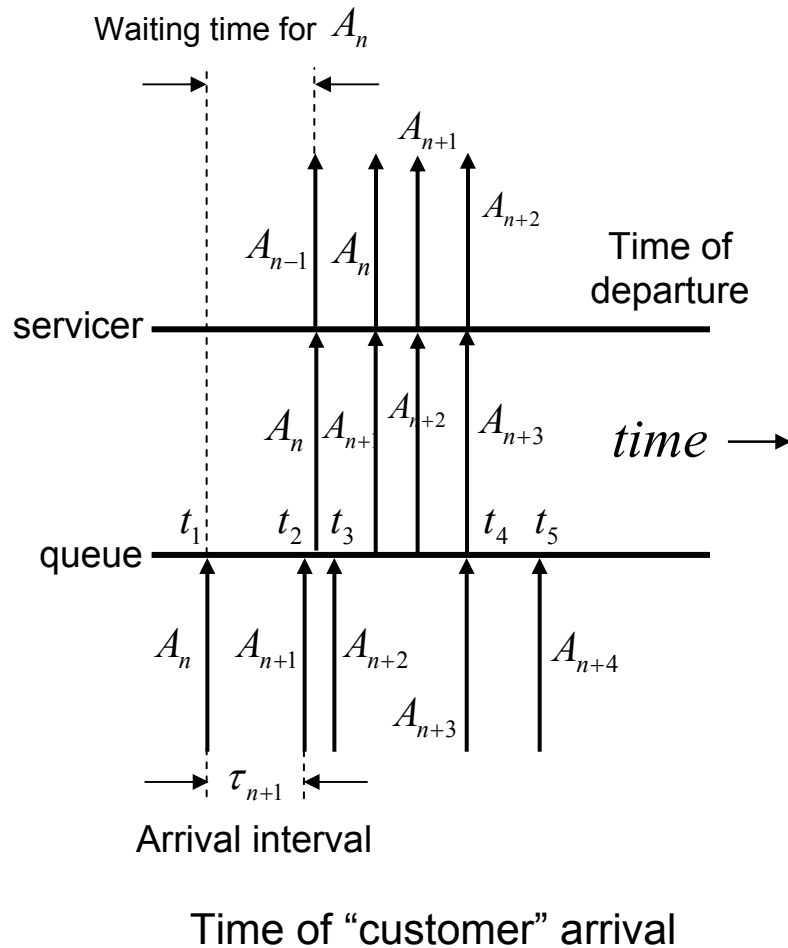
$$\Pr\{X(10) = 2\} = \frac{(1 \times 10)^2}{2!} e^{-1 \times 10} = 0.00277$$

Therefore,

$$\Pr\{X(10) < 3\} = 0.003265$$

Note: at $t=10$, $E[X]=10$ cars, and the probability is 0.1251, much higher than the number 0.003265 above.

Use of Poisson Process in Queuing Analysis



Both the arrival and the departure process, $X(t, \xi)$ and $Y(t, \xi)$, can be modeled as a Poisson process.

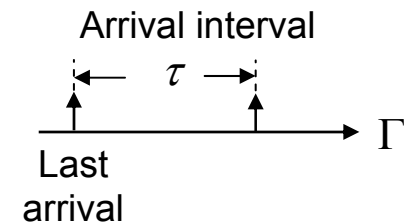
Arrival Interval of A Poisson Process

- Probability of arrival interval, τ , as a random variable is the same as the probability that there is no arrival during that interval.

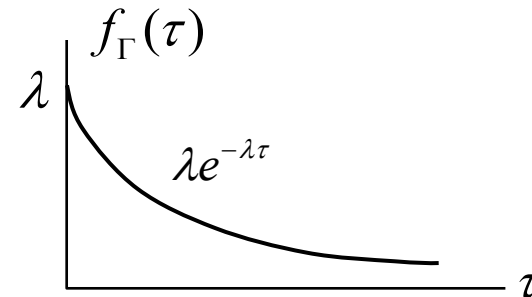
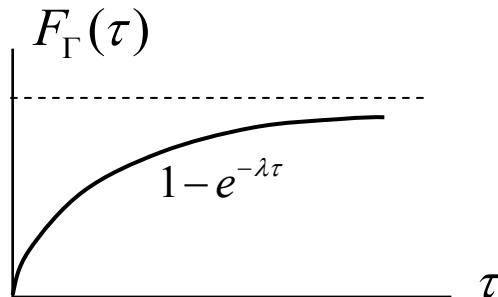
$$\Pr\{X(\tau) = 0\} = \frac{(\lambda\tau)^0}{0!} e^{-\lambda\tau} = e^{-\lambda\tau}$$

$$\begin{aligned} F_{\Gamma}(\tau) &= \Pr\{\text{arrival interval} \leq \tau\} \\ &= 1 - \Pr\{\text{no arrival in } (0, \tau)\} \\ &= 1 - \Pr\{X(\tau) = 0\} = 1 - e^{-\lambda\tau}, \quad \tau \geq 0 \end{aligned}$$

$$f_{\Gamma}(\tau) = \frac{d}{d\tau} F_{\Gamma}(\tau) = \lambda e^{-\lambda\tau}, \quad \tau \geq 0$$



An exponential distribution!

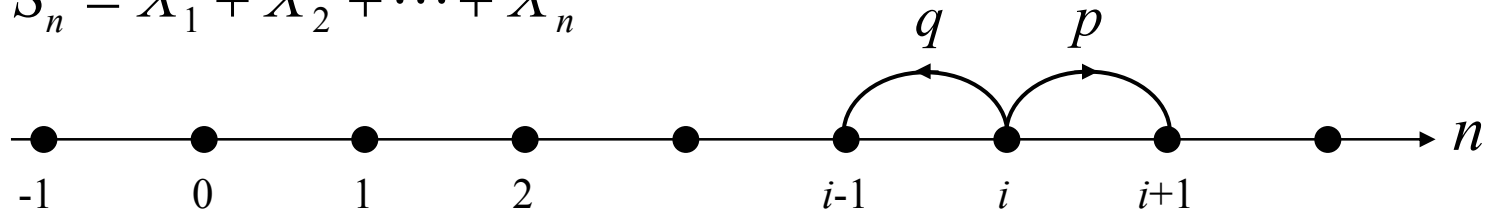


Random Walks

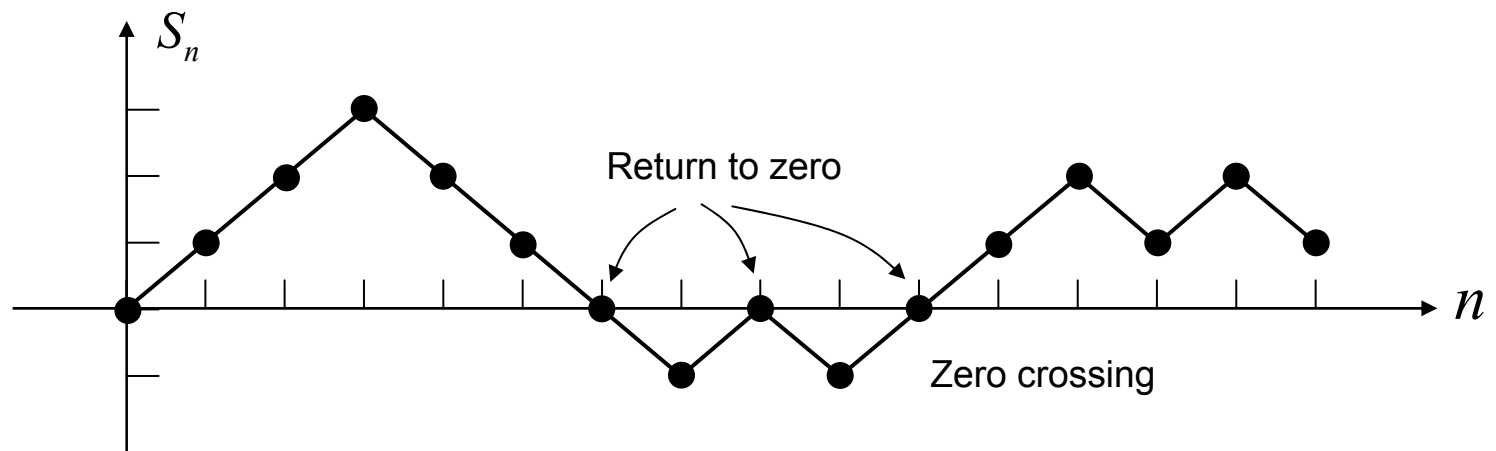
- $X_1, X_2, \dots, X_n, \dots$ are a sequence of independent random variables that assume values $+1$ and -1 with probability p and $q=1-p$, respectively. (The value of 1 is arbitrary subject to scaling.)
- Let $S_n = X_1 + X_2 + \dots + X_n$ be the partial sum with $S_0 = 0$. The partial sum represents the accumulated positive or negative results at the end of n^{th} trial.
- Many real life phenomena can be modeled by a random walk process.
 - The stock value variations of a particular stock;
 - The number of sign changes (or zero crossing, or level crossing) of a signal
 - The motion of gas molecules in a diffusion process
- If $p=q$, it is called a symmetric random walk; otherwise, an asymmetric random walk.

Random Walks

$$S_n = X_1 + X_2 + \dots + X_n$$



Integer state representation: S_n is the state the system is in.



Time function representation: S_n is a function of "time" index.

Random Walks

Let the event $\{S_n = r\}$ be the event “at time n the system is at point r ,” and $p_{n,r} = \Pr\{S_n = r\}$

$$p_{n,r} = \Pr\{S_n = r\} = \binom{n}{k} p^k q^{n-k}$$

where k is the number of $+1$'s in n trials and $n-k$ the number of -1 's, and the net gain, which is r , is

$$r = k - (n - k) = 2k - n \quad \text{or} \quad k = (n + r) / 2$$

Thus,

$$p_{n,r} = \binom{n}{(n+r)/2} p^{(n+r)/2} q^{(n-r)/2}$$

where $(n+r)/2$ is an integer between 0 and n inclusive;
that is, n and r must be both odd or both even at the same time.

Random Walks - Examples

What is the probability that $S_n = r = n$?

$p_{n,n} = \Pr\{S_n = n\} = \Pr\{\text{walk in the same positive direction for } n \text{ times}\}$

$$p_{n,n} = \Pr\{S_n = n\} = \binom{n}{n} p^n q^{n-n} = p^n$$

What is the probability that $S_n = r = 0$?

$$p_{n,0} = \binom{n}{n/2} p^{n/2} q^{n/2} = \binom{n}{n/2} p^{n/2} (1-p)^{n/2}$$

The polynomial coefficient indicates that there are $\binom{n}{n/2} = \frac{n!}{(n/2)!(n/2)!}$ paths that contain equal number of upward and downward steps.

What is the probability that a random walk process will reach a place within 2 steps of the peak, given that n steps have taken place?

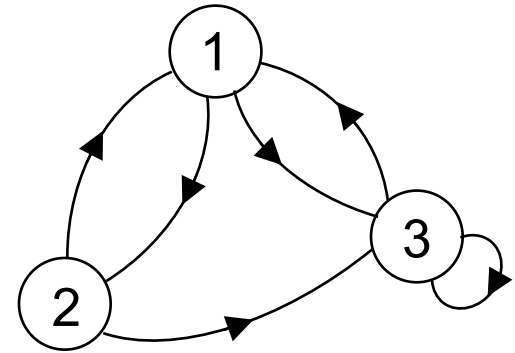
Markov Chain and Processes

- Markov Process Taxonomy
 - By state space
 - Discrete: Markov Chain
 - Continuous: Markov process
 - By time of transition
 - Continuous time
 - Discrete time

$X(t) = b_i$, $i = 1, 2, \dots, N$ depending on the state the system is in.

Markov Property:

$$\Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \dots, X(t_1) = x_1\} \\ = \Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\} \quad \text{where } t_1 < t_2 < \dots < t_n$$



Discrete state

At state 1, $X=b_1$

At state 2, $X=b_2$

At state 3, $X=b_3$

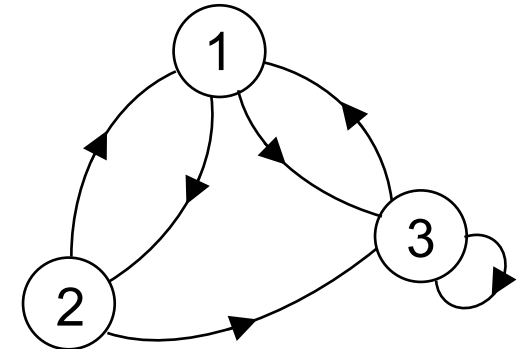
Finite State Markov Chain

- Markov Chain

$$\begin{aligned} & \Pr\{X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \dots, X(t_1) = x_1\} \\ &= \Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \dots, X(t_1) = x_1\} \\ & \quad \bullet \Pr\{X(t_{n-1}) = x_{n-1} \mid X(t_{n-2}) = x_{n-2}, \dots, X(t_1) = x_1\} \bullet \dots \bullet \Pr\{X(t_2) = x_2 \mid X(t_1) = x_1\} \bullet \Pr\{X(t_1) = x_1\} \\ &= \Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\} \Pr\{X(t_{n-1}) = x_{n-1} \mid X(t_{n-2}) = x_{n-2}\} \bullet \dots \bullet \Pr\{X(t_2) = x_2 \mid X(t_1) = x_1\} \bullet \Pr\{X(t_1) = x_1\} \end{aligned}$$

- An N-state (1st order) Markov chain is characterized by an initial state probability array, $\{\Pr[X(t_1)]\}_{\text{all states}}$, and a transition probability matrix \mathbf{A} ,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \dots & \dots & \dots & \dots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix} \quad \sum_{j=1}^N a_{ij} = 1$$



where $a_{ij} = \Pr\{X(t_n) = b_j \mid X(t_{n-1}) = b_i\}$ where $t_{n+1} < t_n$.

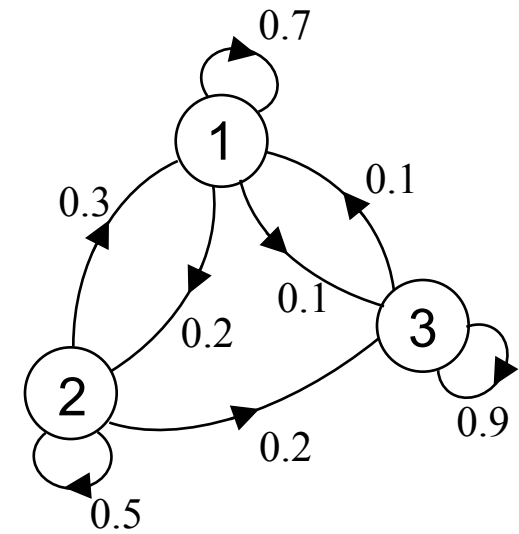
Markov Chain - Example

- A finite-state Markov Chain is governed by the following parameters:

The initial probability vector: $[0.7 \quad 0.2 \quad 0.1]^t$

The state transition matrix:
$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.0 & 0.9 \end{bmatrix}$$

What is the probability that the system goes through the following sequence of states: (1, 3, 3, 1, 3, 1, 2, 1, 1, 3)

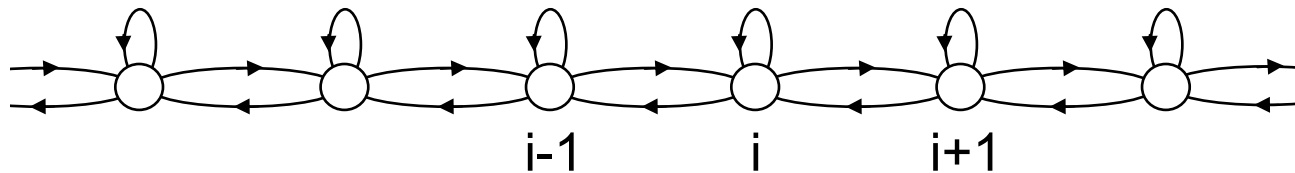


$$\Pr\{(1,3,3,1,3,1,2,1,1,3)\}$$

$$= 0.7 \times 0.1 \times 0.9 \times 0.1 \times 0.1 \times 0.1 \times 0.2 \times 0.3 \times 0.7 \times 0.1 = 2.646 \times 10^{-7}$$

Other Random Processes

- Birth-Death Processes: special case of Markov Process in that the **state** can only move up (one birth) or down (one death) by one or stay the same (no birth or death); however, the value associated with a state can be arbitrary (unlike random walk in which the random variable changes **value** by the same amount, say 1, up or down).



- Renewal Processes: related to random walk but with interest in counting the transitions that takes place as a function of time.