ECE 3075A
Random Signals

Lecture 16
Autocorrelation Functions & Their Properties

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Correlation Functions

- Correlation between two random variables is the expected value of the product of the two random variables. It measures how "coherently" the two random variables behave. If they behave coherently (e.g., when one r.v. is observed to have a high value, the other is likely to have a high value as well), the correlation is high.

  Correlation: $R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dxdy$

  Correlation Coefficient: $\rho_{XY} = E\left[\frac{(X-\overline{X})}{\sigma_X}\frac{(Y-\overline{Y})}{\sigma_Y}\right]$

- A correlation function can be defined in a similar manner between two random processes to measure how "coherently" the two random processes behave. These two processes can refer to the same random process – when this is the case, it is called the autocorrelation function.

Correlation Functions

- The correlation function between two different random processes is called **cross-correlation**, defined as

  $$R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_2 f_{X_1,Y_2}(x_1,y_2)\,dx_1 dy_2$$

  where we use the notation $X(t_1) = X_1$ and $Y(t_2) = Y_2$ of which x_1 and y_2 are a realization, respectively.

- The **autocorrelation function** can be defined in a similar manner, with Y in the above replaced by X:

  $$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1,x_2)\,dx_1 dx_2$$

  Obviously,

  $$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = E[X(t_2)X(t_1)] = R_X(t_2,t_1)$$

  $$R_X(t,t) = E[X(t)X(t)] = E[X^2(t)] \geq 0$$

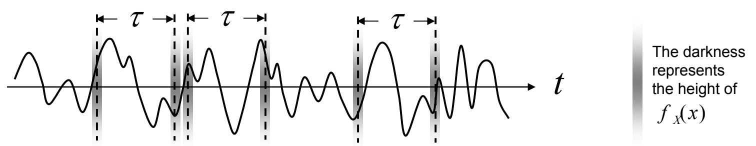
Wide Sense Stationary Processes

$$R_X(t_1,t_2) = R_X(t_1+T, t_2+T) = E[X(t_1+T)X(t_2+T)]$$

Set $T = -t_1 \quad \Rightarrow \quad R_X(t_1,t_2) = R_X(t_1-T, t_2-T) = R_X(0, t_2-t_1)$

For a wide sense stationary process, the autocorrelation function does not depend on the absolute time origin. The first argument 0 is thus arbitrary and the autocorrelation is a function of only the time difference, $t_2 - t_1 = \tau$.

$$R_X(\tau) = R_X(0, t_2-t_1) = E[X(t_1)X(t_1+\tau)] = E[X(t)X(t+\tau)]$$



If we look at any two time instances of a wide sense stationary process, their correlation is only a function of their time difference, no matter where they are. In subsequent discussions, wide sense stationarity is always assumed.

Time Autocorrelation Function

$$\mathbf{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)\,dt = \langle x(t)x(t+\tau)\rangle$$

For an ergodic process, $\mathbf{R}_X(\tau) = R_X(\tau)$

$R_X(0) = E[X(t)X(t)] = E[X^2(t)] =$ mean-square value of the process.

$$\mathbf{R}_X(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)\,dt = \langle x^2(t)\rangle$$

$R_X(0)$ is used as the average or the expected power of a random process.

Example:

In many cases, the noise that gets into a signal is considered I.I.d. with zero mean and variance σ^2. For such processes,

$$R_V(\tau) = E[V(t)V(t+\tau)] = \begin{cases} \sigma^2, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$$

$R_V(0)$ is considered the power of noise.

Interpretation of Autocorrelation

Let $X(t)$ be a zero-mean stationary random process. Form a new random process $Y(t)$ according to $Y(t) = X(t) - \rho X(t+\tau)$.
If the variation in Y is small, we can use $X(t)$ to "predict" $X(t+\tau)$; obtain an early estimate of $X(t+\tau)$ by just dividing the value of $X(t)$ by ρ. But how do we choose ρ such that variation in Y is small?

⟹ We choose ρ to minimize $E[Y^2(t)]$.

$$E[Y^2(t)] = E\{[X(t) - \rho X(t+\tau)]^2\} = E\{X^2(t) - 2\rho X(t)X(t+\tau) + \rho^2 X^2(t+\tau)\}$$

That is, $\sigma_Y^2 = \sigma_X^2 - 2\rho R_X(\tau) + \rho^2\sigma_X^2$

$$\frac{d\sigma_Y^2}{d\rho} = -2R_X(\tau) + 2\rho\sigma_X^2 = 0 \quad\Rightarrow\quad \rho = \frac{R_X(\tau)}{\sigma_X^2}$$ which is the correlation coefficient

Therefore, for simple prediction of $X(t+\tau)$, we use the value $X(t)$ and divide it by the correlation coefficient ρ between $X(t)$ and $X(t+\tau)$. For better prediction, higher order is often needed.

Example 6-1.1

A random process has sample functions of the form on the right:

$$X(t) = \begin{cases} A, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Where A is a random variable uniformly distributed from 0 to 10. Using the basic definition of the autocorrelation function as given by eq. 6-1, find the autocorrelation of the process.

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{XX}(x_1,x_2)\,dx_1dx_2$$

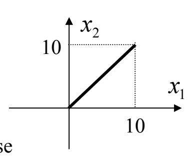
$X(t)$ is a deterministic function because once A is realized, the time function is known.

If $0 \le t_1, t_2 \le 1$,

$$R_X(t_1,t_2) = \int_0^{10} x^2 \frac{1}{10}dx = \frac{1}{10}\frac{x^3}{3}\bigg|_0^{10} = 33.3$$

If t_1 and t_2 are not in (0,1) simultaneously,

If $0 \le t_1, t_2 \le 1$, $f_{X_1,X_2}(x_1,x_2) = 0.1$ on the line $x_1 = x_2$; 0 elsewhere. The (x_1, x_2) corrdinates collapse to a line.



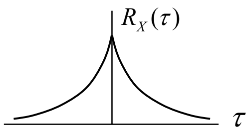
$R_X(t_1,t_2) = 0$, because at least one $x = 0$, making the product and the integral zero.

Example 6-1.2

Define $Z(t) = X(t) + X(t+\tau_1)$ where $X(t)$ is a sample function from a stationary process whose autocorrelation function is

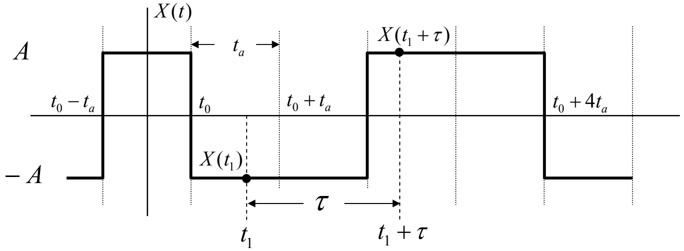
$$R_X(\tau) = \exp(-\tau^2)$$

Write an expression for the autocorrelation function of the random process $Z(t)$.



$$\begin{aligned} R_Z(\tau) &= E[Z(t)Z(t+\tau)] \\ &= E\{[X(t)+X(t+\tau_1)][X(t+\tau)+X(t+\tau+\tau_1)]\} \\ &= E[X(t)X(t+\tau) + X(t+\tau_1)X(t+\tau) \\ &\quad + X(t)X(t+\tau+\tau_1) + X(t+\tau_1)X(t+\tau+\tau_1)] \\ &= R_X(\tau) + R_X(\tau-\tau_1) + R_X(\tau+\tau_1) + R_X(\tau) \\ &= 2\exp(-\tau^2) + \exp[-(\tau-\tau_1)^2] + \exp[-(\tau+\tau_1)^2] \end{aligned}$$

Autocorrelation of a Binary Process



- A discrete (equi-probable at $\pm A$), stationary, zero-mean process.
- State change clocked at t_a interval with arbitrary starting time, t_0 ; that is, t_0 is considered a random variable uniformly distributed over (0, t_a).
- $X(t)$ in one interval is statistically independent from $X(t)$ in another interval.
- The process is very common in data communications and digital computers.

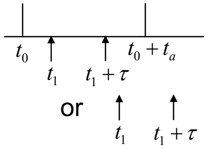
Autocorrelation of a Binary Process

$X(t_1)$ and $X(t_2)$ are independent if $|t_1 - t_2| = |\tau| > t_a$.

Therefore, due to the fact that the process is stationary, zero-mean,

$$R_X(\tau) = E[X(t)X(t+\tau)] = E[X(t)]E[X(t+\tau)] = 0, \quad |\tau| > t_a$$

When $|\tau| < t_a$, t_1 and $t_2 = t_1 + \tau$ may or may not be in the same interval, depending on the value of t_0.



$\Pr\{t_1$ and $t_1 + \tau,\ \tau > 0,$ are in the same interval $\}$

$$= \Pr\{t_1 + \tau - t_a < t_0 \le t_1\} = \frac{1}{t_a}[t_1 - (t_1 + \tau - t_a)] = \frac{t_a - \tau}{t_a}$$

$\Pr\{t_1$ and $t_1 + \tau,\ \tau < 0,$ are in the same interval $\}$

$$= \Pr\{t_1 - t_a < t_0 \le t_1 + \tau\} = \frac{1}{t_a}[t_1 + \tau - (t_1 - t_a)] = \frac{t_a + \tau}{t_a}$$

$$\Pr\{t_1 \text{ and } t_1 + \tau \text{ are in the same interval}\} = \frac{t_a - |\tau|}{t_a}$$

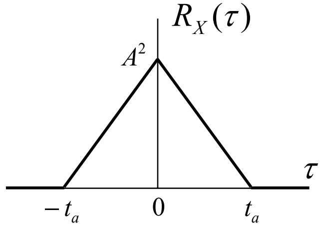
Autocorrelation of a Binary Process

When t_1 and t_2 are in the same interval, the product of X_1 and X_2 is always A^2 ; when they are not, X_1 and X_2 are independent with zero mean and thus zero correlation. Hence,

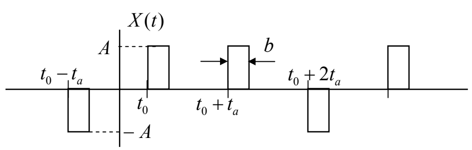
$$R_X(\tau) = \begin{cases} A^2\left[\dfrac{t_a - |\tau|}{t_a}\right] = A^2\left[1 - \dfrac{|\tau|}{t_a}\right], & |\tau| \le t_a \\ 0, & |\tau| > t_a \end{cases}$$

Remarks:

- When the two time instances are close to each other, the two corresponding r.v.s are likely to have the same value;
- When they are apart far enough, it is equally probable that they'll have the same value as they'll have the opposite value;
- At $\tau = 0$, the autocorrelation is the same as the mean square value, representing the power of the signal.



Example 6-2.2



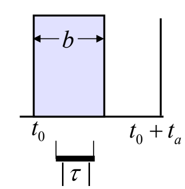
The process is the same as the binary process previously discussed except that it now does not have a full duty cycle but only b/t_a .

The product of $X(t_1)$ and $X(t_2)$ is zero if $|t_1 - t_2| = |\tau| > b$.

When $|\tau| < b$, t_1 and $t_2 = t_1 + \tau$ may or may not be in the same interval.

The range that the center of the time-bar ($|\tau|$ wide) can be in for it to be totally in the shaded area is $\left(t_0 + \frac{|\tau|}{2}, t_0 + b - \frac{|\tau|}{2}\right)$

This range has a width of $b - |\tau|$.



Therefore,

$$\Pr\{t_1 \text{ and } t_1 + \tau \text{ are in the same active duty interval}\} = \frac{b - |\tau|}{t_a}$$

$$R_X(\tau) = \begin{cases} A^2\left[\dfrac{b - |\tau|}{t_a}\right] = A^2\dfrac{b}{t_a}\left[1 - \dfrac{|\tau|}{b}\right], & |\tau| \le b \\ 0, & |\tau| > b \end{cases}$$

b/t_a is called the duty cycle.

Properties of Autocorrelation Functions

- $R_X(0) = \overline{X^2} \geq 0$
- Symmetry: $R_X(\tau) = R_X(-\tau)$

$$R_X(\tau) = E[X(t)X(t+\tau) = E[X(t-\tau)X(t)] = R_X(-\tau)$$

- $|R_X(\tau)| \leq R_X(0)$

$$E[(X_1 \pm X_2)^2] = E[X_1^2 \pm 2X_1X_2 + X_2^2] \geq 0$$

$$E[X_1^2 + X_2^2] = 2R_X(0) \geq |E[2X_1X_2]| = 2|R_X(\tau)|$$

- If $X(t)$ has a constant component, say, $X(t) = A + V(t)$, where $V(t)$ has zero mean, the autocorrelation function has a constant component.

$$E\{[A+V(t)][A+V(t+\tau)]\} = E[A^2 + AV(t) + AV(t+\tau) + V(t)V(t+\tau)]$$

$$= A^2 + AE[V(t)] + AE[V(t+\tau)] + E[V(t)V(t+\tau)] = A^2 + R_V(\tau)$$

Properties of Autocorrelation Functions

- If $X(t)$ has a periodic component, then the autocorrelation function has a periodic component.

$$X(t) = A\cos(\omega t + \Theta)$$

where A and ω are constant and Θ a r.v. uniformly distributed over (0,2π); i.e. $f_\Theta(\theta) = (2\pi)^{-1}, 0 \leq \theta < 2\pi; = 0,$ elsewhere.

$$R_X(\tau) = E[A\cos(\omega t + \Theta)A\cos(\omega t + \omega\tau + \Theta)]$$

$$= E\left[\frac{A^2}{2}\cos(2\omega t + \omega\tau + 2\Theta) + \frac{A^2}{2}\cos(\omega\tau)\right]$$

$$= \frac{A^2}{2}\cos(\omega\tau) + \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\cos(2\omega t + \omega\tau + 2\theta)\,d\theta = \frac{A^2}{2}\cos(\omega\tau)$$

- If $X(t)$ is ergodic and zero-mean, and has no periodic components,

$$\lim_{|\tau|\to\infty} R_X(\tau) = 0$$

That is, time samples far apart tend to behave statistically independently.

Properties of Autocorrelation Function

- Autocorrelation functions cannot have arbitrary shape – they must correspond to some power spectrum which must be non-negative over the entire frequency range. More discussions later.

$$\mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau = \text{power spectrum of } X(t)$$

$$\mathcal{F}[R_X(\tau)] = S_X(\omega) \geq 0 \quad \text{for all } \omega$$

Example: An ergodic random process has an autocorrelation function of the form

$$R_X(\tau) = \frac{4\tau^2 + 6}{\tau^2 + 1}$$

Find the mean-square value, mean value, and variance of the process.

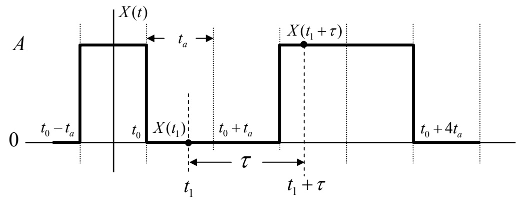
$$R_X(\tau) = \frac{4\tau^2 + 6}{\tau^2 + 1} = 4 + \frac{2}{\tau^2 + 1} \qquad \overline{X^2} = R_X(0) = 6$$

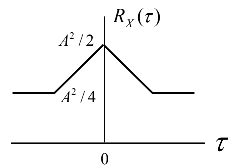
$$\overline{X}^2 = 4 \Rightarrow \overline{X} = \pm 2 \qquad \sigma_X^2 = \overline{X^2} - (\overline{X})^2 = 6 - 4 = 2$$

Other Examples of Autocorrelation Functions

1. Binary process with uniformly spaced switching intervals – see previous discussion.
2. Binary process with uniformly spaced switching intervals and non-zero mean.
3. Binary process with randomly spaced switching times – the telegraph process.
4. Bandpass filtered signals – discussion will take place with introduction of power spectrum.

Binary Process with Non-zero Mean





$X(t) = \frac{A}{2} + \frac{1}{2}X'(t)$ where $X'(t)$ is the binary process previously discussed.

$$R_X(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{4} + \frac{A}{4}E[X'(t)] + \frac{A}{4}E[X'(t+\tau)] + \frac{1}{4}E[X'(t)X'(t+\tau)]$$

$$= \frac{A^2}{4} + \frac{1}{4}R_{X'}(\tau) = \frac{A^2}{4} + \frac{1}{4}R_{X'}(\tau)$$

$$R_{X'}(\tau) = \begin{cases} A^2\left[\frac{t_a - |\tau|}{t_a}\right] = A^2\left[1 - \frac{|\tau|}{t_a}\right], & |\tau| \le t_a \\ 0, & |\tau| > t_a \end{cases}$$

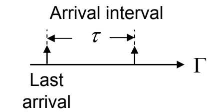
Therefore,

$$R_X(\tau) = \begin{cases} \frac{A^2}{4} + \frac{A^2}{4}\left[1 - \frac{|\tau|}{t_a}\right], & |\tau| \le t_a \\ \frac{A^2}{4}, & |\tau| > t_a \end{cases}$$

Arrival Interval of A Poisson Process

- Probability of arrival interval, τ, as a random variable is the same as the probability that there is no arrival during that interval.

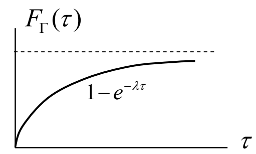
$$\Pr\{X(\tau)=0\} = \frac{(\lambda\tau)^0}{0!}e^{-\lambda\tau} = e^{-\lambda\tau}$$

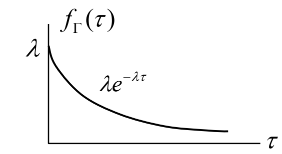


$$F_\Gamma(\tau) = \Pr\{\text{arrival interval} \le \tau\}$$
$$= 1 - \Pr\{\text{no arrival in } (0,\tau)\}$$
$$= 1 - \Pr\{X(\tau)=0\} = 1 - e^{-\lambda\tau}, \quad \tau \ge 0$$

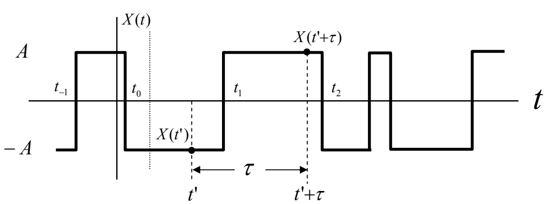
$$f_\Gamma(\tau) = \frac{d}{d\tau}F_\Gamma(\tau) = \lambda e^{-\lambda\tau}, \quad \tau \ge 0$$

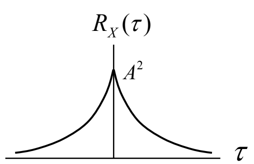
An exponential distribution!





Binary Process with Random Switching Times





- The switching times $\{t_i\}_{i=-\infty}^{\infty}$ occur as Poisson arrivals – a point process.
- When τ is less than the switching interval, the correlation between $X(t')$ and $X(t'+\tau)$ has a value A^2.

$$\Pr\{t' \text{ and } t'+\tau \text{ fall within a switching interval}\}$$
$$= \Pr\{\text{no switching or arrival in } (0,|\tau|)\} = \exp(-\lambda|\tau|)$$

Therefore, $R_X(\tau) = A^2\exp(-\lambda|\tau|)$

λ is the rate of switching (average number of switching per unit time).