ECE 3075A

Random Signals

Lecture 17

Autocorrelation Functions & Properties

School of Electrical and Computer Engineering
Georgia Institute of Technology
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Interpretation of Autocorrelation

Let $X(t)$ be a zero-mean stationary random process. Form a new random process $Y(t)$ according to $Y(t) = X(t) - \rho X(t+\tau)$.
If the variation in Y is small, we can use $X(t)$ to "predict" $X(t+\tau)$; obtain an early estimate of $X(t+\tau)$ by just dividing the value of $X(t)$ by ρ. But how do we choose ρ such that variation in Y is small?

$\Longrightarrow$ We choose ρ to minimize $E[Y^2(t)]$.

$$E[Y^2(t)] = E\{[X(t) - \rho X(t+\tau)]^2\} = E\{X^2(t) - 2\rho X(t)X(t+\tau) + \rho^2 X^2(t+\tau)\}$$

That is, $\sigma_Y^2 = \sigma_X^2 - 2\rho R_X(\tau) + \rho^2 \sigma_X^2$

$$\frac{d\sigma_Y^2}{d\rho} = -2R_X(\tau) + 2\rho\sigma_X^2 = 0 \quad \Rightarrow \quad \rho = \frac{R_X(\tau)}{\sigma_X^2}$$

the correlation coefficient between $X(t)$ and $X(t+\tau)$

Therefore, for simple prediction of $X(t+\tau)$, we use the value $X(t)$ and divide it by the correlation coefficient ρ between $X(t)$ and $X(t+\tau)$. For better prediction, higher order is often needed.

Example 6-1.1

A random process has sample functions of the form on the right:

$$X(t) = \begin{cases} A, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Where A is a random variable uniformly distributed from 0 to 10. Using the basic definition of the autocorrelation function as given by eq. 6-1, find the autocorrelation of the process.

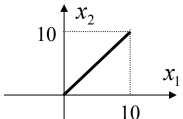
$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{XX}(x_1,x_2)\,dx_1 dx_2$$

$X(t)$ is a deterministic function because once A is realized, the time function is known.

If $0 \le t_1, t_2 \le 1$,

$$R_X(t_1,t_2) = \int_0^{10} x^2 \frac{1}{10} dx = \frac{1}{10}\frac{x^3}{3}\Big|_0^{10} = 33.3$$

If $0 \le t_1, t_2 \le 1$, $f_{X_1,X_2}(x_1,x_2) = 0.1$ on the line $x_1 = x_2$; 0 elsewhere. The (x_1, x_2) corrdinates collapse to a line.



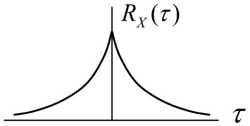
If t_1 and t_2 are not in (0,1) simultaneously, $R_X(t_1,t_2) = 0$, because at least one $X = 0$, making the product and the integral zero.

Example 6-1.2

Define $Z(t) = X(t) + X(t+\tau_1)$ where $X(t)$ is a sample function from a stationary process whose autocorrelation function is

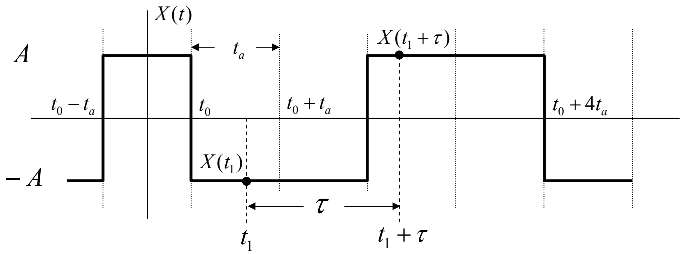
$$R_X(\tau) = \exp(-\tau^2)$$

Write an expression for the autocorrelation function of the random process $Z(t)$.



$$\begin{aligned}
R_Z(\tau) &= E[Z(t)Z(t+\tau)] \\
&= E\{[X(t) + X(t+\tau_1)][X(t+\tau) + X(t+\tau+\tau_1)]\} \\
&= E[X(t)X(t+\tau) + X(t+\tau_1)X(t+\tau) \\
&\quad + X(t)X(t+\tau+\tau_1) + X(t+\tau_1)X(t+\tau+\tau_1)] \\
&= R_X(\tau) + R_X(\tau-\tau_1) + R_X(\tau+\tau_1) + R_X(\tau) \\
&= 2\exp(-\tau^2) + \exp[-(\tau-\tau_1)^2] + \exp[-(\tau+\tau_1)^2]
\end{aligned}$$

Autocorrelation of a Binary Process



- A discrete (equi-probable at $\pm A$), stationary, zero-mean process.
- State change clocked at t_a interval with arbitrary starting time, t_0 ; that is, t_0 is considered a random variable uniformly distributed over (0, t_a).
- $X(t)$ in one interval is statistically independent from $X(t)$ in another interval.
- The process is very common in data communications and digital computers.

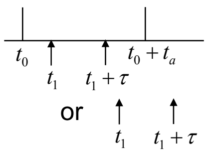
Autocorrelation of a Binary Process

$X(t_1)$ and $X(t_2)$ are independent if $|t_1 - t_2| = |\tau| > t_a$.

Therefore, due to the fact that the process is stationary, zero-mean,

$$R_X(\tau) = E[X(t)X(t+\tau)] = E[X(t)]E[X(t+\tau)] = 0, \quad |\tau| > t_a$$

When $|\tau| < t_a$, t_1 and $t_2 = t_1 + \tau$ may or may not be in the same interval, depending on the value of t_0.



$$\Pr\{t_1 \text{ and } t_1+\tau,\ \tau>0, \text{ are in the same interval}\}$$

$$= \Pr\{t_1+\tau-t_a < t_0 \le t_1\} = \frac{1}{t_a}[t_1 - (t_1+\tau-t_a)] = \frac{t_a - \tau}{t_a}$$

$$\Pr\{t_1 \text{ and } t_1+\tau,\ \tau<0, \text{ are in the same interval}\}$$

$$= \Pr\{t_1 - t_a < t_0 \le t_1+\tau\} = \frac{1}{t_a}[t_1+\tau-(t_1-t_a)] = \frac{t_a+\tau}{t_a}$$

$$\Pr\{t_1 \text{ and } t_1+\tau \text{ are in the same interval}\} = \frac{t_a - |\tau|}{t_a}$$

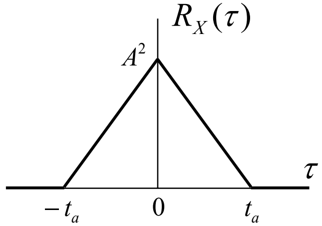
Autocorrelation of a Binary Process

When t_1 and t_2 are in the same interval, the product of X_1 and X_2 is always A^2 ; when they are not, X_1 and X_2 are independent with zero mean and thus zero correlation. Hence,

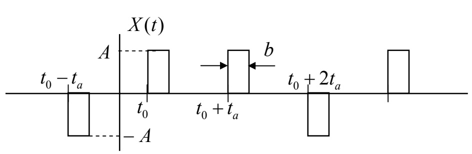
$$R_X(\tau) = \begin{cases} A^2\left[\dfrac{t_a - |\tau|}{t_a}\right] = A^2\left[1 - \dfrac{|\tau|}{t_a}\right], & |\tau| \le t_a \\ 0, & |\tau| > t_a \end{cases}$$

Remarks:

- When the two time instances are close to each other, the two corresponding r.v.s are likely to have the same value;
- When they are apart far enough, it is equally probable that they'll have the same value as they'll have the opposite value;
- At $\tau = 0$, the autocorrelation is the same as the mean square value, representing the power of the signal.



Example 6-2.2



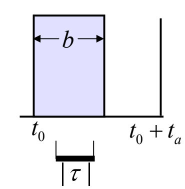
The process is the same as the binary process previously discussed except that it now does not have a full duty cycle but only b/t_a .

The product of $X(t_1)$ and $X(t_2)$ is zero if $|t_1 - t_2| = |\tau| > b$.

When $|\tau| < b$, t_1 and $t_2 = t_1 + \tau$ may or may not be in the same interval.

The range that the center of the time-bar ($|\tau|$ wide) can be in for it to be totally in the shaded area is $\left(t_0 + \frac{|\tau|}{2}, t_0 + b - \frac{|\tau|}{2}\right)$

This range has a width of $b - |\tau|$.



Therefore,

$$\Pr\{t_1 \text{ and } t_1+\tau \text{ are in the same active duty interval}\} = \frac{b - |\tau|}{t_a}$$

$$R_X(\tau) = \begin{cases} A^2\left[\dfrac{b-|\tau|}{t_a}\right] = A^2\dfrac{b}{t_a}\left[1 - \dfrac{|\tau|}{b}\right], & |\tau| \le b \\ 0, & |\tau| > b \end{cases}$$

b/t_a is called the duty cycle.

Properties of Autocorrelation Functions

- $R_X(0) = \overline{X^2} \geq 0$
- Symmetry: $R_X(\tau) = R_X(-\tau)$

$$R_X(\tau) = E[X(t)X(t+\tau) = E[X(t-\tau)X(t)] = R_X(-\tau)$$

- $|R_X(\tau)| \leq R_X(0)$

$$E[(X_1 \pm X_2)^2] = E[X_1^2 \pm 2X_1X_2 + X_2^2] \geq 0$$

$$E[X_1^2 + X_2^2] = 2R_X(0) \geq |E[2X_1X_2]| = 2|R_X(\tau)|$$

- If $X(t)$ has a constant component, say, $X(t) = A + V(t)$, where $V(t)$ has zero mean, the autocorrelation function has a constant component.

$$E\{[A+V(t)][A+V(t+\tau)]\} = E[A^2 + AV(t) + AV(t+\tau) + V(t)V(t+\tau)]$$

$$= A^2 + AE[V(t)] + AE[V(t+\tau)] + E[V(t)V(t+\tau)] = A^2 + R_V(\tau)$$

Properties of Autocorrelation Functions

- If $X(t)$ has a periodic component, then the autocorrelation function has a periodic component.

$$X(t) = A\cos(\omega t + \Theta)$$

where A and ω are constant and Θ a r.v. uniformly distributed over (0,2π); i.e. $f_\Theta(\theta) = (2\pi)^{-1}, 0 \leq \theta < 2\pi; = 0,$ elsewhere.

$$R_X(\tau) = E[A\cos(\omega t + \Theta)A\cos(\omega t + \omega\tau + \Theta)]$$

$$= E\left[\frac{A^2}{2}\cos(2\omega t + \omega\tau + 2\Theta) + \frac{A^2}{2}\cos(\omega\tau)\right]$$

$$= \frac{A^2}{2}\cos(\omega\tau) + \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\cos(2\omega t + \omega\tau + 2\theta)\,d\theta = \frac{A^2}{2}\cos(\omega\tau)$$

- If $X(t)$ is ergodic and zero-mean, and has no periodic components,

$$\lim_{|\tau|\to\infty} R_X(\tau) = 0$$

That is, time samples far apart tend to behave statistically independently.

Properties of Autocorrelation Function

- Autocorrelation functions cannot have arbitrary shape – they must correspond to some power spectrum which must be non-negative over the entire frequency range. More discussions later.

$$\mathbf{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau = \text{power spectrum of } X(t)$$

$$\mathbf{F}[R_X(\tau)] = S_X(\omega) \geq 0 \quad \text{for all } \omega$$

$\mathbf{F}$ is the Fourier transform

Example: An ergodic random process has an autocorrelation function of the form

$$R_X(\tau) = \frac{4\tau^2 + 6}{\tau^2 + 1}$$

Find the mean-square value, mean value, and variance of the process.

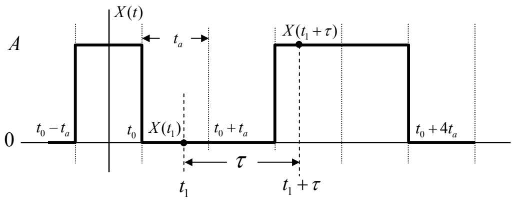
$$R_X(\tau) = \frac{4\tau^2 + 6}{\tau^2 + 1} = 4 + \frac{2}{\tau^2 + 1} \qquad \overline{X^2} = R_X(0) = 6$$

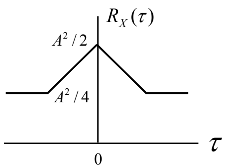
$$\overline{X}^2 = 4 \quad \Rightarrow \quad \overline{X} = \pm 2 \qquad \sigma_X^2 = \overline{X^2} - (\overline{X})^2 = 6 - 4 = 2$$

Other Examples of Autocorrelation Functions

1. Binary process with uniformly spaced switching intervals – see previous discussion.
2. Binary process with uniformly spaced switching intervals and non-zero mean.
3. Binary process with randomly spaced switching times – the telegraph process.
4. Bandpass filtered signals – discussion will take place with introduction of power spectrum.

Binary Process with Non-zero Mean





$X(t) = \frac{A}{2} + \frac{1}{2}X'(t)$ where $X'(t)$ is the binary process previously discussed.

$$R_X(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{4} + \frac{A}{4}E[X'(t)] + \frac{A}{4}E[X'(t+\tau)] + \frac{1}{4}E[X'(t)X'(t+\tau)]$$

$$= \frac{A^2}{4} + \frac{1}{4}R_{X'}(\tau) = \frac{A^2}{4} + \frac{1}{4}R_{X'}(\tau)$$

$$R_{X'}(\tau) = \begin{cases} A^2\left[\frac{t_a - |\tau|}{t_a}\right] = A^2\left[1 - \frac{|\tau|}{t_a}\right], & |\tau| \le t_a \\ 0, & |\tau| > t_a \end{cases}$$

Therefore,

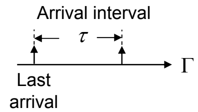
$$R_X(\tau) = \begin{cases} \frac{A^2}{4} + \frac{A^2}{4}\left[1 - \frac{|\tau|}{t_a}\right], & |\tau| \le t_a \\ \frac{A^2}{4}, & |\tau| > t_a \end{cases}$$

Arrival Interval of A Poisson Process

- Probability of arrival interval, τ, as a random variable is the same as the probability that there is no arrival during that interval.

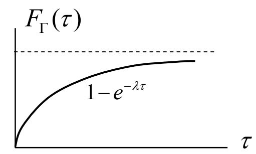
$$\Pr\{X(\tau) = 0\} = \frac{(\lambda\tau)^0}{0!}e^{-\lambda\tau} = e^{-\lambda\tau}$$

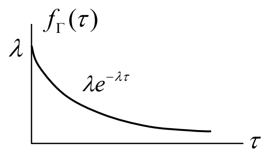
$$F_\Gamma(\tau) = \Pr\{\text{arrival interval} \le \tau\}$$
$$= 1 - \Pr\{\text{no arrival in } (0,\tau)\}$$
$$= 1 - \Pr\{X(\tau) = 0\} = 1 - e^{-\lambda\tau}, \quad \tau \ge 0$$



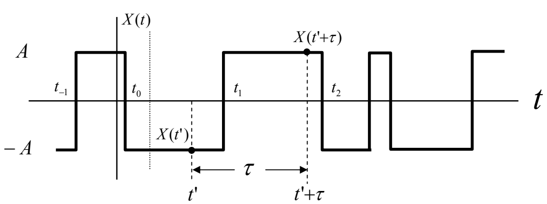
$$f_\Gamma(\tau) = \frac{d}{d\tau}F_\Gamma(\tau) = \lambda e^{-\lambda\tau}, \quad \tau \ge 0$$

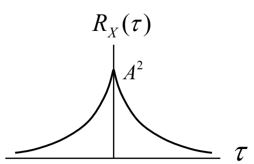
An exponential distribution!





Binary Process with Random Switching Times





- The switching times $\{t_i\}_{i=-\infty}^{\infty}$ occur as Poisson arrivals – a point process.
- When τ is less than the switching interval, the correlation between $X(t')$ and $X(t'+\tau)$ has a value A^2.

$$\Pr\{t' \text{ and } t'+\tau \text{ fall within a switching interval}\}$$
$$= \Pr\{\text{no switching or arrival in } (0,|\tau|)\} = \exp(-\lambda|\tau|)$$

Therefore, $R_X(\tau) = A^2\exp(-\lambda|\tau|)$

λ is the rate of switching (average number of switching per unit time).