ECE 3075A
Random Signals

Lecture 18
Crosscorrelation Functions, Discrete-time Processes

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Crosscorrelation Functions

- The correlation function between two different random processes is called a crosscorrelation function.
- The crosscorrelation function measures how "coherently" two processes behave together, at various points in time.
- Crosscorrelation functions are very much of interest in system anaylsis; for example, in studying the relationship between the input and the output of an electronic system, the relationship between the interest rate and various market indices, etc.

Jointly W-S Stationary Processes

- A process $X(t)$ is wide sense stationary if
  $E[X(t)] = \overline{X} =$ constant (independent of time), and
  $E[X(t)X(t+\tau)] = R_{XX}(\tau)$, a function of only the time difference.
- Two processes $X(t)$ and $Y(t)$ are jointly wide sense stationary if
  1. Both $X(t)$ and $Y(t)$ are wide sense stationary, and
  2. $R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau)$,
     a function of only the time difference.

Recall the original definition:

$$R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} x_1 y_2 f_{X_1Y_2}(x_1,y_2)\,dy_2$$

where we have used the notation $X_1 = X(t_1), Y_2 = Y(t_2)$

Crosscorrelation of W-S Stationary Processes

- $X(t)$ and $Y(t)$ are jointly stationary in wide sense.
- X_1 and Y_2 are two random variables,
  $X_1 = X(t_1)$ and $Y_2 = Y(t_1+\tau)$
- The crosscorrelation function is defined as

$$R_{XY}(\tau) = E[X_1Y_2] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} x_1 y_2 f_{X_1Y_2}(x_1,y_2)\,dy_2$$

- The temporal order is of significance:

$$R_{YX}(\tau) = E[Y_1X_2] = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} y_1 x_2 f_{Y_1X_2}(y_1,x_2)\,dx_2$$

  In the above, $Y_1 = Y(t_1)$ and $X_2 = X(t_1+\tau)$.

Let $t_1 = t_2 - \tau$, it then follows that $t_2 = t_1 + \tau$ and $R_{XY}(\tau) = R_{YX}(-\tau)$.

Time Crosscorrelation Functions

- Define $\mathbf{R}_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)\,dt$

$$\mathbf{R}_{YX}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y(t)x(t+\tau)\,dt$$

- If the processes are jointly ergodic,

$$\mathbf{R}_{XY}(\tau) = R_{XY}(\tau), \text{ and } \mathbf{R}_{YX}(\tau) = R_{YX}(\tau)$$

Example:
Two jointly random processes are of the form
$X(t) = 2\cos(5t+\Theta)$ and $Y(t) = 10\sin(5t+\Theta)$
Where Θ is a random variable unformly distributed in (0, 2π). Find the crosscorrelation of the two processes

$$R_{XY}(\tau) = E[2\cos(5t+\Theta)\,10\sin(5t+5\tau+\Theta)]$$
$$= 10\sin(5\tau) + E[10\sin(10t+5\tau+2\Theta)] = 10\sin(5\tau)$$

Properties of Crosscorrelation Functions

- $R_{XY}(0)$ and $R_{YX}(0)$ only measures the correlation of the two processes at synchronous points.
- $R_{XY}(0) = R_{YX}(0); \quad R_{XY}(\tau) = R_{YX}(-\tau)$
- $|R_{XY}(\tau)| \le [R_X(0)R_Y(0)]^{1/2}$
- The maximum of a crosscorrelation function can occur anywhere, not necessarily at $\tau = 0$.
- If the two random processes are independent,

$$R_{XY}(\tau) = E[X_1Y_2] = E[X_1]E[Y_2] = \overline{X}_1\overline{Y}_2 = R_{YX}(\tau)$$

And if any of the process has zero mean, the crosscorrelation function vanishes everywhere.

Applications of Crosscorrelation Functions

$$Z(t) = X(t) \pm Y(t)$$
$$Z_1 = Z(t_1) = X(t_1) \pm Y(t_1) = X_1 \pm Y_1$$
$$Z_2 = Z(t_1+\tau) = X(t_1+\tau) \pm Y(t_1+\tau) = X_2 \pm Y_2$$
$$R_Z(\tau) = E[Z_1Z_2] = E[(X_1 \pm Y_1)(X_2 \pm Y_2)]$$
$$= E[X_1X_2 + Y_1Y_2 \pm X_1Y_2 \pm Y_1X_2]$$
$$= R_X(\tau) + R_Y(\tau) \pm R_{XY}(\tau) \pm R_{YX}(\tau)$$

The autocorrelation function of the sum is the sum of all the autocorrelation functions plus the sum of all the crosscorrelation functions.

If the two random processes are statistically independent and one of them has zero mean, the cross terms varnish, resulting in "the autocorrelation function of the sum is the sum of the autocorrelation functions."

$$R_{XY}(\tau) = E[X_1Y_2] = E[X_1]E[Y_2] = 0 \quad \text{if } \overline{X}_1 = 0 \text{ or } \overline{Y}_2 = 0.$$

Sinusoid Plus Noise

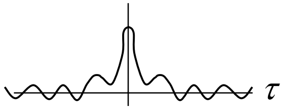
Let $X(t) = A\cos(\omega t + \Theta)$ where Θ is a random variable uniformly distributed over $(0, 2\pi)$. $R_X(\tau) = \frac{1}{2}A^2\cos\omega\tau$

Let $V(t)$ be a zero mean noise process, statistically independent of $X(t)$, the signal, with autocorrelation function:

$$R_V(\tau) = B^2 e^{-\alpha|\tau|}$$

The observed process is $Z(t) = A\cos(\omega t + \Theta) + V(t)$ which has an autocorrelation function:

$$R_Z(\tau) = \frac{1}{2}A^2\cos\omega\tau + B^2 e^{-\alpha|\tau|}$$



Note that $R_V(\tau) = B^2 e^{-\alpha|\tau|} \to 0, \text{ as } \tau \to \infty.$

It is thus possible to recover a sinusoid from noise contanimation as long as we measure the autocorrelation at sufficiently long time lags.

Use of Crosscorrelation - Example

A radar sends out a signal $X(t)$ which is returned from a target, after a round-trip delay of τ_1. The receiver receives $Y(t)$, which includes the delayed and attenuated signal $X(t-\tau_1)$ and the noise $V(t)$:

$$Y(t) = aX(t-\tau_1) + V(t) \quad \text{where } a < 1$$

We use the crosscorrelation function between X and Y to detect the presence of the returned signal in Y.

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t+\tau)] \\ &= E[aX(t)X(t+\tau-\tau_1) + X(t)V(t+\tau)] \\ &= aR_X(\tau-\tau_1) + R_{XV}(\tau) = aR_X(\tau-\tau_1) \end{aligned}$$

because X and V are statistically independent and V has zero mean.

$R_{XY}(\tau) = aR_X(\tau-\tau_1)$ attains maximum at $\tau = \tau_1$, from which the distance to the target can be calculated.

Example – 6-8.1

A random process has the form $X(t)=A$ in which A is a random variable with mean value 5 and variance 10. This random process can be observed only in the presence of independent noise $V(t)$ having an auto correlation function of

$$R_V(\tau) = 10\exp(-2|\tau|)$$

1. Find the autocorrelation function of the sum of these two processes;
2. If the autocorrelation function of the sum is observed, find the value of τ at which the autocorrealtion function is within 1% of its value at $\tau = \infty$.

$$\begin{aligned} Y(t) &= X(t) + V(t) \\ R_Y(\tau) &= E[Y(t)Y(t+\tau)] \\ &= E[X(t)X(t+\tau)] + E[X(t)V(t+\tau)] + E[V(t)X(t+\tau)] + E[V(t)V(t+\tau)] \\ &= E[A^2] + R_V(\tau) = \sigma_A^2 + \overline{A}^2 + R_V(\tau) = 35 + 10\exp(-2|\tau|) \end{aligned}$$

$$R_Y(\tau) = 35 + 10\exp(-2|\tau|), \qquad R_Y(\infty) = 35$$

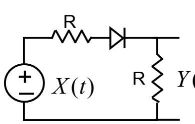
$$R_Y(\tau) = 35 + 10\exp(-2|\tau|) = 35.35 \ \Rightarrow \ \exp(-2|\tau|) = 0.035$$

Therefore, at $\tau = \pm(-0.5\ln 0.035) = \pm 1.6762$ $R_Y(\tau)$ is within 1% of $R_Y(\infty)$.

Example – 6-8.2

A random binary process (see Section 6-2) has amplitude of ±12 and $t_a = 0.01$. It is applied to the half-wave rectifier circuit as shown on the right.

Find the autocorrelation of the output, $R_Y(\tau)$
Find the crosscorrelation function $R_{XY}(\tau)$
Find the crosscorrelation function $R_{YX}(\tau)$.



The amplitude of $Y(t)$ is either 0 (when $X(t)$ is -12) or half of $X(t)$ when it is +12, i.e. +6. The mean of $Y(t)$ is thus 3.

$Y(t) = \overline{Y} + W(t)$ where $W(t)$ is a zero-mean binary process with amplitude ± 3.

$$R_Y(\tau) = \overline{Y}^2 + R_W(\tau) = \begin{cases} 9 + 9\left(1 - \dfrac{|\tau|}{0.01}\right), & |\tau| \le 0.01 \\ 9, & |\tau| > 0.01 \end{cases}$$

For crosscorrelation:

If $|\tau| > t_a$, $X(t)$ and $Y(t)$ are independent and $X(t)$ has zero mean, thus $R_{XY}(\tau) = 0$.

If $|\tau| \le t_a$, $Y(t)$ is half $(=+6)$ of $X(t)$ $(=+12)$ half of the time; the half of the time, $X(t)$ is negative and $Y(t)$ is 0. Thus $R_{XY}(\tau) = \dfrac{1}{2} \times 12 \times 6 \times \left(1 - \dfrac{|\tau|}{0.01}\right) = 36\left(1 - \dfrac{|\tau|}{0.01}\right)$.

Discrete-Time Sequences as R. P.

$X(t)$ is observed at equally spaced time instants. Let

$X_1 = X(\Delta t), X_2 = X(2\Delta t), \cdots, X_N = X(N\Delta t)$, then

$$\hat{\overline{X}} = \frac{1}{N}\sum_{i=1}^{N} X_i \ \text{ which is a discrete-time version of } \ \hat{\overline{X}} = \frac{1}{T}\int_{t=0}^{T} X(t)\,dt.$$

$$E[\hat{\overline{X}}] = E\left[\frac{1}{N}\sum_{i=1}^{N} X_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[X_i] = \frac{1}{N}\sum_{i=1}^{N}\overline{X} = \overline{X}$$

$$E[(\hat{\overline{X}})^2] = E\left[\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j\right] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E[X_i X_j]$$

If these time-sampled random variables are statistically independent,

$$E[X_i X_j] = \overline{X^2} \ \text{ if } i = j; \ = (\overline{X})^2 \ \text{ if } i \ne j.$$

Thus, $E[(\hat{\overline{X}})^2] = \dfrac{1}{N^2}[N\overline{X^2} + (N^2 - N)\overline{X}^2]$

Discrete-Time Processes

$$E[(\hat{\bar{X}})^2] = \frac{1}{N^2}[N\overline{X^2} + (N^2 - N)\bar{X}^2] = \frac{\overline{X^2}}{N} + \left(1 - \frac{1}{N}\right)\bar{X}^2 = \frac{\sigma_X^2}{N} + \bar{X}^2$$

$$\mathrm{var}(\hat{\bar{X}}) = E[(\hat{\bar{X}})^2] - \bar{X}^2 = \frac{\sigma_X^2}{N}$$

The variance is *1/N* of the variance of the process. Therefore, the more data used in time average for estimating the mean of the process, the better the result. This fact is similar to the sample mean when the statistics of a population was discussed.

Similarly, the sample variance is a biased estimate of the variance of the process. We eliminate the bias by using the following as an unbiased estimate of the variance of the process:

$$\hat{\sigma}_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \hat{\bar{X}})^2 = \frac{1}{N-1}\sum_{i=1}^{N}X_i^2 - \frac{N}{N-1}(\hat{\bar{X}})^2$$

Autocorrelation of Discrete-Time Sequences

Use of a "partial time autocorrelation function" as an estimate of the autocorrelation function of the process:

$$\hat{R}_X(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} x(t)x(t+\tau)\,dt$$

Implemented in discrete time,

$$\hat{R}_X(n\Delta t) = \frac{1}{N-n+1}\sum_{k=0}^{N-n} x_k x_{k+n} \qquad n = 0, 1, 2, \cdots, M, M << N$$

Quality of estimate: Unbiased

$$E[\hat{R}_X(n\Delta t)] = E\left[\frac{1}{N-n+1}\sum_{k=0}^{N-n} X_k X_{k+n}\right] = \frac{1}{N-n+1}\sum_{k=0}^{N-n} E[X_k X_{k+n}] = \frac{1}{N-n+1}\sum_{k=0}^{N-n} R_X(n\Delta t) = R_X(n\Delta t)$$

However, for a smaller mean-square error, often used is

$$\hat{R}_X(n\Delta t) = \frac{1}{N+1}\sum_{k=0}^{N-n} x_k x_{k+n} \qquad n = 0, 1, 2, \cdots, M, M << N$$

Autocorrelation Matrices

Use vector notations,

$$\mathbf{X} = [X(t_1) \quad X(t_2) \quad \cdots \quad X(t_N)]^t = [X_1 \quad X_2 \quad \cdots \quad X_N]^t$$

Then,

$$\mathbf{R_X} = E[\mathbf{X}\mathbf{X}^t] = \begin{bmatrix} E[X(t_1)X(t_1)] & E[X(t_1)X(t_2)] & \cdots & E[X(t_1)X(t_N)] \\ E[X(t_2)X(t_1)] & E[X(t_2)X(t_2)] & \cdots & E[X(t_2)X(t_N)] \\ \cdots & \cdots & \cdots & \cdots \\ E[X(t_N)X(t_1)] & E[X(t_N)X(t_2)] & \cdots & E[X(t_N)X(t_N)] \end{bmatrix}$$

$$= \begin{bmatrix} R_X(t_1,t_1) & R_X(t_1,t_2) & \cdots & R_X(t_1,t_N) \\ R_X(t_2,t_1) & R_X(t_2,t_2) & \cdots & R_X(t_2,t_N) \\ \cdots & \cdots & \cdots & \cdots \\ R_X(t_N,t_1) & R_X(t_N,t_2) & \cdots & R_X(t_N,t_N) \end{bmatrix}$$

Often we are dealing with wide sense stationary processes sampled at discrete times. Then,

$$\mathbf{R_X} = \begin{bmatrix} R_X[0] & R_X[\Delta t] & \cdots & R_X[(N-1)\Delta t] \\ R_X[\Delta t] & R_X[0] & \cdots & R_X[(N-2)\Delta t] \\ \cdots & \cdots & \cdots & \cdots \\ R_X[(N-1)\Delta t] & R_X[(N-2)\Delta t] & \cdots & R_X[0] \end{bmatrix}$$

Covariance Matrices

Also we often are dealing with normal (Gaussian) processes. (Recall joint distributions of Gaussian random variables.) The covariance matrix is a more direct quantity in the definition of joint distributions than the autocorrelation matrix. $\mathbf{C}$ is the same as $\mathbf{\Lambda}$ in the text.

$$\mathbf{C}_X = E[(\mathbf{X} - \bar{\mathbf{X}})(\mathbf{X} - \bar{\mathbf{X}})^t] = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1N}\sigma_1\sigma_N \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_{2N}\sigma_2\sigma_N \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{N1}\sigma_N\sigma_1 & \rho_{N2}\sigma_N\sigma_2 & \cdots & \sigma_N^2 \end{bmatrix}$$

$$\mathbf{C}_X = E[(\mathbf{X} - \bar{\mathbf{X}})(\mathbf{X} - \bar{\mathbf{X}})^t] = \mathbf{R}_X - (\bar{\mathbf{X}})(\bar{\mathbf{X}})^t$$

For a wide sense stationary process,

$$\left.\begin{matrix} \sigma_i^2 = \sigma_j^2 = \sigma^2 \\ \rho_{ij} = \rho_{|i-j|} \end{matrix}\right\} \quad \text{for } i, j = 1, 2, ..., N$$

$$\mathbf{C}_X = \sigma^2 \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{N-1} \\ \rho_1 & 1 & \cdots & \rho_{N-2} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{N-1} & \rho_{N-2} & \cdots & 1 \end{bmatrix}$$

A Toeplitz matrix

Revisit to W-S Gaussian Processes

$$\mathbf{X} = [X(t_1) \quad X(t_2) \quad \cdots \quad X(t_N)]^t = [X_1 \quad X_2 \quad \cdots \quad X_N]^t$$

$$f_{\mathbf{X}}(x_1, x_2, \cdots, x_N) = \frac{\left|\mathbf{C}_X^{-1}\right|^{1/2}}{(2\pi)^{N/2}} \exp\left\{-\frac{(\mathbf{X}-\overline{\mathbf{X}})^t \mathbf{C}_X^{-1} (\mathbf{X}-\overline{\mathbf{X}})}{2}\right\}$$

$$\mathbf{X}-\overline{\mathbf{X}} = \begin{bmatrix} x_1 - \overline{X}_1 \\ x_2 - \overline{X}_2 \\ \vdots \\ x_N - \overline{X}_N \end{bmatrix} = \begin{bmatrix} x_1 - \overline{X} \\ x_2 - \overline{X} \\ \vdots \\ x_N - \overline{X} \end{bmatrix} \qquad \mathbf{C}_X = \sigma^2 \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{N-1} \\ \rho_1 & 1 & \cdots & \rho_{N-2} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{N-1} & \rho_{N-2} & \cdots & 1 \end{bmatrix}$$

When N = 2,

$$\mathbf{C}_X = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \qquad \mathbf{C}_X^{-1} = \frac{1}{(1-\rho^2)\sigma^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \qquad \left|\mathbf{C}_X^{-1}\right| = [\sigma^4(1-\rho^2)]^{-1}$$

$$f_{\mathbf{X}}(x_1, x_2) = \frac{1}{2\pi\sigma^2\sqrt{(1-\rho^2)}} \exp\left\{-\frac{(x_1-\overline{X})^2 - 2\rho(x_1-\overline{X})(x_2-\overline{X}) + (x_2-\overline{X})^2}{2\sigma^2(1-\rho^2)}\right\}$$

Gaussian Process - Example

A wide sense stationary Gaussian process has a mean of $\overline{X} = 4$ and autocorrelation function $R_{XX}(\tau) = 25e^{-3|\tau|}$

Specify the joint density functions for three random variables $X(t_i), i = 1, 2, 3,$ at times $t_i = t_0 + [(i-1)/2]$ with t_0 a constant.

$t_j - t_i = (j-i)/2$ for i and $j = 1, 2, 3$

$$R_{XX}(\tau) = 25e^{-3|j-i|/2}$$

$$C_{XX}(t_j - t_i)$$
$$= R_{XX}(t_j - t_i) - E[X(t_i)]E[X(t_j)]$$
$$= 25e^{-3|j-i|/2} - 16$$

Practical insight: the autocorrelation of most if not all real signals will eventually go to zero (not necessarily monotonically though) as τ grows.

$$\mathbf{C}_{XX} = \begin{bmatrix} (25-16) & (25e^{-3/2}-16) & (25e^{-6/2}-16) \\ (25e^{-3/2}-16) & (25-16) & (25e^{-3/2}-16) \\ (25e^{-6/2}-16) & (25e^{-3/2}-16) & (25-16) \end{bmatrix}$$

Crosscorrelation Matrix

$$\mathbf{X}(t) = [X_1(t) \quad X_2(t) \quad \cdots \quad X_N(t)]^t$$
$$\mathbf{Y}(t) = [Y_1(t) \quad Y_2(t) \quad \cdots \quad Y_N(t)]^t$$

$\mathbf{X}$ and $\mathbf{Y}$ can be considered as signals transmitted and received (at time t) by multiple antennas at N different points.

We can define a crosscorrelation matrix between $\mathbf{X}$ and $\mathbf{Y}$,

$$\mathbf{R}_{\mathbf{XY}}(\tau) = E[\mathbf{X}(t)\mathbf{Y}^t(t+\tau)] = \begin{bmatrix} R_{11}(\tau) & R_{12}(\tau) & \cdots & R_{1N}(\tau) \\ R_{21}(\tau) & R_{22}(\tau) & \cdots & R_{2N}(\tau) \\ \cdots & \cdots & \cdots & \cdots \\ R_{N1}(\tau) & R_{N2}(\tau) & \cdots & R_{NN}(\tau) \end{bmatrix}$$

Sometimes, $\mathbf{X}$ and $\mathbf{Y}$ may not have the same number of dimensions. The crosscorrelation matrix in that case then is not a square matrix.