

ECE 3075A
Random Signals

Lecture 19
Spectral Density

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Spectrum

- Frequency-domain representation of a time function.
- Based on the principle of decomposing a time function into sum of (complex) sinusoids.
- The mathematical tool for doing such decomposition is the Fourier transform:

$$F_X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_X(\omega)e^{j\omega t} d\omega$$

Fourier transform

Inverse Fourier transform

$F_X(\omega)$ (a complex quantity) is the relative magnitude and phase of a steady-state (complex) sinusoid of frequency ω . When $F_X(\omega)$ at all the frequencies are summed together, we have the original time function or signal, as seen in the inverse transform.

Amplitude Spectrum and Phase Spectrum

$$F_X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad F_X(\omega) = |F_X(\omega)| e^{j\angle F_X(\omega)}$$

$$\ln F_X(\omega) = \ln |F_X(\omega)| + j\angle F_X(\omega)$$

$\text{Re}\{\ln F_X(\omega)\} = \ln |F_X(\omega)|$, log-magnitude spectrum

$\text{Im}\{\ln F_X(\omega)\} = \angle F_X(\omega)$, phase spectrum

$|F_X(\omega)|$ is the amplitude of the sinusoid of frequency ω , and as a function of ω , is called the **amplitude or magnitude density** of the signal $x(t)$. It gives an indication how the energy of $x(t)$ is distributed across the frequency range, possibly from $-\infty$ to ∞ .

When the function is a random process, in order for $|F_X(\omega)|$ to exist, or $X(t)$ to be Fourier transformable, $X(t)$ has to be absolute-summable: $\int_{-\infty}^{\infty} |X(t)| dt < \infty$

A wide sense stationary process with non-zero mean is not absolutely summable. Why?

Fourier Transform of A Random Process

- Sample functions of a wide-sense stationary process are usually not absolutely summable and therefore are not Fourier transformable.

Absolute summability requires: $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

- Need to modify the process so that the transform exists; use a truncated process:

$$X_T(t) = \begin{cases} X(t), & |t| \leq T < \infty \\ 0, & |t| > T \end{cases} \quad \Rightarrow \quad \int_{-\infty}^{\infty} |X_T(t)| dt < \infty$$

$$\int_{-\infty}^{\infty} |X_T(t)|^2 dt < \infty$$

Then its Fourier transform is

$$F_{X_T}(\omega) = \int_{-\infty}^{\infty} X_T(t)e^{-j\omega t} dt$$

This is an intermediate step for us to derive the power density spectrum of a random process.

Power Density Spectrum

$$F_{X_T}(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t} dt = \int_{-T}^T X(t) e^{-j\omega t} dt$$

From Parseval's theorem, (signal energy calculated in the time domain is the same as that calculated in the frequency domain)

$$\int_{-T}^T X_T^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_{X_T}(\omega)|^2 d\omega$$

In terms of power:
$$\frac{1}{2T} \int_{-T}^T X_T^2(t) dt = \frac{1}{4\pi T} \int_{-\infty}^{\infty} |F_{X_T}(\omega)|^2 d\omega$$

Taking expectation:
$$E\left[\frac{1}{2T} \int_{-T}^T X_T^2(t) dt\right] = E\left\{\frac{1}{4\pi T} \int_{-\infty}^{\infty} |F_{X_T}(\omega)|^2 d\omega\right\}$$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[X_T^2(t)] dt = \lim_{T \rightarrow \infty} \frac{1}{4\pi T} \int_{-\infty}^{\infty} E[|F_{X_T}(\omega)|^2] d\omega$$

Since $E[X_T^2(t)] = \overline{X^2}$ for $-T \leq t \leq T$,
$$\overline{X^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{E[|F_{X_T}(\omega)|^2]}{2T} d\omega$$

Define spectral density:
$$S_X(\omega) = \lim_{T \rightarrow \infty} \frac{E[|F_{X_T}(\omega)|^2]}{2T}$$

Spectral Density

$$S_X(\omega) = \lim_{T \rightarrow \infty} \frac{E[|F_{X_T}(\omega)|^2]}{2T} \quad \overline{X^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$

Since $S_X(\omega)$ is an average over time, it is thus usually called a **power density spectrum**. When $S_X(\omega)$ is integrated over the entire frequency range, we obtain the average power of the signal, which is equal to the mean-square value of the wide-sense stationary process.

Example:
$$S_X(\omega) = \frac{2a}{\omega^2 + a^2} \quad S_X(0) = \frac{2a}{0^2 + a^2} = \frac{2}{a}$$

$$\overline{X^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{\omega^2 + a^2} d\omega = \frac{1}{2\pi} \left[\frac{2a}{a} \tan^{-1}\left(\frac{\omega}{a}\right) \right]_{-\infty}^{\infty} = \frac{2}{2\pi} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = 1$$

Useful integral:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{\omega^2 + a^2} d\omega = \frac{1}{\pi} \left[\tan^{-1}\left(\frac{\omega}{a}\right) \right]_p \quad \int_p^q \frac{2a}{(2\pi f)^2 + a^2} df = \frac{1}{\pi} \left[\tan^{-1}\left(\frac{2\pi f}{a}\right) \right]_p$$

ω in radian/s f in cycle/s or Hz

Example 7-2.2

A stationary random process has a two-sided spectral density given by

$$S_X(\omega) = \frac{24}{\omega^2 + 16} \quad \text{V}^2/\text{Hz}$$

- Find the mean-square value of the process;
- Find the power of the process in the frequency range of ± 1 Hz centered at the origin.

$$\overline{X^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{3 \times 8}{\omega^2 + 4^2} d\omega = \frac{3}{\pi} \left[\tan^{-1}\left(\frac{\omega}{4}\right) \right]_{-\infty}^{\infty} = \frac{3}{\pi} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = 3 \quad (\text{V}^2)$$

$$\begin{aligned} \overline{X_1^2} &= \int_{-1}^1 S_X(f) df = \int_{-1}^1 \frac{3 \times 8}{(2\pi f)^2 + 4^2} df = \frac{3}{\pi} \left[\tan^{-1}\left(\frac{2\pi f}{4}\right) \right]_{-1}^1 \\ &= \frac{3}{\pi} \left[\tan^{-1}\left(\frac{\pi}{2}\right) + \tan^{-1}\left(\frac{\pi}{2}\right) \right] = 1.91728 \quad (\text{V}^2) \end{aligned}$$

Properties of Spectral Density

- Spectral density is a real, non-negative and even function of frequency (ω or f).
- Since it is an even function of the frequency, a rational spectral density of the form

$$S_X(\omega) = \frac{S_0(\omega^{2n} + a_{2n-2}\omega^{2n-2} + \dots + a_2\omega^2 + a_0)}{\omega^{2m} + b_{2m-2}\omega^{2m-2} + \dots + b_2\omega^2 + b_0}$$

contains only even powers of ω .

- White noise is a process whose spectral density is a constant over the entire frequency range, i.e.,

$$S_X(\omega) = S_0 \quad \text{for all } \omega$$

(just like a “white” light contains all colored lights with equal intensity). When the marginal distribution is Gaussian, we call it a **Gaussian white noise**.

Spectral Density of Constant or Periodic Signals

Consider a process $X(t) = A + B \cos(2\pi f_0 t + \Theta)$
 where A, B and f_0 are constant and Θ is a random variable uniformly distributed over $(0, 2\pi)$.

Let $X_T(t)$ be a truncated version of $X(t)$ over $(-T, T)$.

$$F_{X_T}(f) = \int_{-T}^T [A + B \cos(2\pi f_0 t + \Theta)] e^{-j2\pi f t} dt$$

$$\mathbf{F}[X(t)] = \int_{-\infty}^{\infty} [A + B \cos(2\pi f_0 t + \Theta)] e^{-j2\pi f t} dt$$

$$= A\delta(f) + (B/2)[\delta(f + f_0)e^{-j\Theta} + \delta(f - f_0)e^{j\Theta}]$$

$$F_{X_T}(f) = 2T \text{sinc}(2Tf) * \mathbf{F}[X(t)]$$

$$= 2AT \text{sinc}(2Tf) + BT \{ \text{sinc}[2(f + f_0)T] e^{-j\Theta} + \text{sinc}[2(f - f_0)T] e^{j\Theta} \}$$

$$|F_{X_T}(f)|^2 = F_{X_T}(f) F_{X_T}^*(f)$$

$$= (2AT)^2 \text{sinc}^2(2Tf) + (BT)^2 \{ \text{sinc}^2[2(f + f_0)T] + \text{sinc}^2[2(f - f_0)T] \}$$

$$+ C(f)e^{-j\Theta} + C(-f)e^{j\Theta} + D(f)e^{-j2\Theta} + D(-f)e^{j2\Theta}$$

$$S_X(\omega) = \lim_{T \rightarrow \infty} \frac{E[|F_{X_T}(\omega)|^2]}{2T}$$

Spectral Density of Constant or Periodic Signals

$E[|F_{X_T}(f)|^2]$
 $= (2AT)^2 \text{sinc}^2(2Tf) + (BT)^2 \{ \text{sinc}^2[2(f + f_0)T] + \text{sinc}^2[2(f - f_0)T] \}$
 because $E_\Theta[G(f)e^{-j\Theta}] = 0$ and $E_\Theta[H(f)e^{-j2\Theta}] = 0$ for Θ uniformly distributed in $(0, 2\pi)$

$$S_X(f) = \lim_{T \rightarrow \infty} \frac{E[|F_{X_T}(f)|^2]}{2T}$$

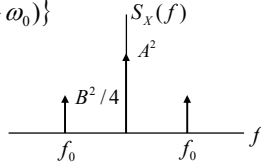
$$= \lim_{T \rightarrow \infty} \{ A^2 2T \text{sinc}^2(2Tf) + (B^2/4) (2T \text{sinc}^2[2(f + f_0)T] + 2T \text{sinc}^2[2(f - f_0)T]) \}$$

But, $\lim_{T \rightarrow \infty} 2T \text{sinc}^2(2Tf) = \delta(f)$

Therefore, $S_X(f) = A^2 \delta(f) + (B^2/4) \{ \delta(f + f_0) + \delta(f - f_0) \}$

And, $S_X(\omega) = 2\pi A^2 \delta(\omega) + (\pi B^2/2) \{ \delta(\omega + \omega_0) + \delta(\omega - \omega_0) \}$

The spectral density thus consists of three spikes (delta function) at DC (with height A^2) and at $\pm f_0$ (with height $B^2/4$), respectively.



Mean-Square Value and Total Power

$X(t) = A + B \cos(\omega_0 t + \Theta)$ Θ uniformly distributed in $(0, 2\pi)$

$$S_X(\omega) = 2\pi A^2 \delta(\omega) + (\pi B^2/2) \{ \delta(\omega + \omega_0) + \delta(\omega - \omega_0) \}$$

Total power:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ 2\pi A^2 \delta(\omega) + (\pi B^2/2) [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \} d\omega$$

$$= A^2 + \frac{B^2}{4} + \frac{B^2}{4} = A^2 + \frac{B^2}{2}$$

Mean-Square Value of the process

$$E_\Theta[X^2(t)] = E_\Theta \{ [A + B \cos(\omega_0 t + \Theta)]^2 \} = E_\Theta \{ A^2 + 2AB \cos(\omega_0 t + \Theta) + B^2 \cos^2(\omega_0 t + \Theta) \}$$

$$= A^2 + E_\Theta \{ 2AB \cos(\omega_0 t + \Theta) \} + B^2 E_\Theta \{ \cos^2(\omega_0 t + \Theta) \}$$

$$E_\Theta \{ 2AB \cos(\omega_0 t + \Theta) \} = 0 \text{ for } \Theta \sim \mathcal{U}(0, 2\pi)$$

$$E_\Theta \{ \cos^2(\omega_0 t + \Theta) \} = E_\Theta \left\{ \frac{\cos(2\omega_0 t + 2\Theta) + 1}{2} \right\} = \frac{1}{2}$$

Therefore, $E_\Theta[X^2(t)] = \overline{X^2} = A^2 + \frac{B^2}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$

Example 7-3.1

A stationary random process has a spectral density of the form

$$S_X(f) = 4\delta(f) + 18\delta(f + 8) + 18\delta(f - 8)$$

- Find the discrete frequencies present;
- Find the mean value of the process;
- Find the variance of the process.

$$S_X(f) = A^2 \delta(f) + (B^2/4) \{ \delta(f + f_0) + \delta(f - f_0) \}$$

Frequencies present are 0, +8 and -8 Hz respectively.

The mean value of the process corresponds to the amplitude of the DC (i.e., zero-frequency) component which has a power spectral density of 4. Therefore, the mean is ± 2 .

The mean-square value is $4 + 18 + 18 = 40$

Thus, the variance is $\sigma^2 = \overline{X^2} - (\overline{X})^2 = 40 - 4 = 36$

Time Derivative of A Random Process

Let $\dot{X}(t) = dX(t) / dt$

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_X(\omega) e^{j\omega t} d\omega$$

$$\dot{X}(t) = \frac{dX(t)}{dt} = \frac{d}{dt} \frac{1}{2\pi} \int_{-\infty}^{\infty} F_X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega F_X(\omega) e^{j\omega t} d\omega$$

$$\mathbf{E}[\dot{X}(t)] = j\omega F_X(\omega)$$

$$S_{\dot{X}}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|j\omega F_{X_T}(\omega)(-j\omega)F_{X_T}(-\omega)|]}{2T} = \omega^2 \lim_{T \rightarrow \infty} \frac{E[|F_{X_T}(\omega)|^2]}{2T} = \omega^2 S_X(\omega)$$

In case $S_X(\omega)$ does not drop off more rapidly than $1/\omega^2$ as $\omega \rightarrow \infty$, then the mean - square value of the derivative process will become infinite. The process is then said to be non - differentiable.