ECE 3075A
Random Signals

Lecture 19
Spectral Density

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Spectrum

- Frequency-domain representation of a time function.
- Based on the principle of decomposing a time function into sum of (complex) sinusoids.
- The mathematical tool for doing such decomposition is the Fourier transform:

$$F_X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad\qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_X(\omega) e^{j\omega t} d\omega$$

Fourier transform   Inverse Fourier transform

$F_X(\omega)$ (a complex quantity) is the relative magnitude and phase of a steady-state (complex) sinusoid of frequency ω. When $F_X(\omega)$ at all the frequencies are summed together, we have the original time function or signal, as seen in the inverse transform.

Amplitude Spectrum and Phase Spectrum

$$F_X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad F_X(\omega) = |F_X(\omega)| e^{j\angle F_X(\omega)}$$

$$\ln F_X(\omega) = \ln |F_X(\omega)| + j\angle F_X(\omega)$$

$$\mathrm{Re}\{\ln F_X(\omega)\} = \ln |F_X(\omega)|, \quad \text{log-magnitude spectrum}$$

$$\mathrm{Im}\{\ln F_X(\omega)\} = \angle F_X(\omega), \quad \text{phase spectrum}$$

$|F_X(\omega)|$ is the amplitude of the sinusoid of frequency ω, and as a function of ω, is called the **amplitude or magnitude density** of the signal $x(t)$. It gives an indication how the energy of $x(t)$ is distributed across the frequency range, possibly from -∞ to ∞.

When the function is a random process, in order for $|F_X(\omega)|$ to exist, or $X(t)$ to be Fourier transformable, $X(t)$ has to be absolute-summable: $\int_{-\infty}^{\infty} |X(t)| dt < \infty$

A wide sense stationary process with non-zero mean is not absolutely summable. Why?

Fourier Transform of A Random Process

- Sample functions of a wide-sense stationary process are usually not absolutely summable and therefore are not Fourier transformable.

Absolute summability requires: $\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$

- Need to modify the process so that the transform exists; use a truncated process:

$$X_T(t) = \begin{cases} X(t), & |t| \le T < \infty \\ 0, & |t| > T \end{cases} \quad \Rightarrow \quad \begin{aligned} &\int_{-\infty}^{\infty} |X_T(t)|\,dt < \infty \\ &\int_{-\infty}^{\infty} |X_T(t)|^2\,dt < \infty \end{aligned}$$

Then its Fourier transform is

$$F_{X_T}(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t}\,dt$$

This is an intermediate step for us to derive the power density spectrum of a random process.

Power Density Spectrum

$$F_{X_T}(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t} dt = \int_{-T}^{T} X(t) e^{-j\omega t} dt$$

From Parseval's theorem, (signal energy calculated in the time domain is the same as that calculated in the frequency domain)

$$\int_{-T}^{T} X_T^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} | F_{X_T}(\omega) |^2 \, d\omega$$

In terms of power: $$\frac{1}{2T} \int_{-T}^{T} X_T^2(t) dt = \frac{1}{4\pi T} \int_{-\infty}^{\infty} | F_{X_T}(\omega) |^2 \, d\omega$$

Taking expectation: $$E\left[\frac{1}{2T} \int_{-T}^{T} X_T^2(t) dt \right] = E\left\{ \frac{1}{4\pi T} \int_{-\infty}^{\infty} | F_{X_T}(\omega) |^2 \, d\omega \right\}$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X_T^2(t)] dt = \lim_{T\to\infty} \frac{1}{4\pi T} \int_{-\infty}^{\infty} E[| F_{X_T}(\omega) |^2] d\omega$$

Since $E[X_T^2(t)] = \overline{X^2}$ for $-T \le t \le T$, $\overline{X^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[| F_{X_T}(\omega) |^2]}{2T} d\omega$

Define spectral density: $$S_X(\omega) = \lim_{T\to\infty} \frac{E[| F_{X_T}(\omega) |^2]}{2T}$$

Spectral Density

$$S_X(\omega) = \lim_{T\to\infty} \frac{E[|F_{X_T}(\omega)|^2]}{2T}$$

$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega$$

Since $S_X(\omega)$ is an average over time, it is thus usually called a **power density spectrum**. When $S_X(\omega)$ is integrated over the entire frequency range, we obtain the average power of the signal, which is equal to the mean-square value of the wide-sense stationary process.

Example: $S_X(\omega) = \dfrac{2a}{\omega^2 + a^2}$ $\quad S_X(0) = \dfrac{2a}{0^2 + a^2} = \dfrac{2}{a}$

$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{\omega^2 + a^2}d\omega = \frac{1}{2\pi}\left[\frac{2a}{a}\tan^{-1}\left(\frac{\omega}{a}\right)\right]_{-\infty}^{\infty} = \frac{2}{2\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 1$$

Useful integral:

$$\frac{1}{2\pi}\int_p^q \frac{2a}{\omega^2 + a^2}d\omega = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{\omega}{a}\right)\right]_p^q$$

ω in radian/s

$$\int_p^q \frac{2a}{(2\pi f)^2 + a^2}df = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{2\pi f}{a}\right)\right]_p^q$$

f in cycle/s or Hz

Example 7-2.2

A stationary random process has a two-sided spectral density given by

$$S_X(\omega) = \frac{24}{\omega^2 + 16} \quad \mathrm{V^2/Hz}$$

a. Find the mean-square value of the process;
b. Find the power of the process in the frequency range of ± 1 Hz centered at the origin.

$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{3\times 8}{\omega^2 + 4^2}d\omega = \frac{3}{\pi}\left[\tan^{-1}\left(\frac{\omega}{4}\right)\right]_{-\infty}^{\infty} = \frac{3}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 3 \quad (\mathrm{V}^2)$$

$$\overline{X_1^2} = \int_{-1}^{1} S_X(f)df = \int_{-1}^{1} \frac{3\times 8}{(2\pi f)^2 + 4^2}df = \frac{3}{\pi}\left[\tan^{-1}\left(\frac{2\pi f}{4}\right)\right]_{-1}^{1}$$

$$= \frac{3}{\pi}\left[\tan^{-1}\left(\frac{\pi}{2}\right) + \tan^{-1}\left(\frac{\pi}{2}\right)\right] = 1.91728 \quad (\mathrm{V}^2)$$

Properties of Spectral Density

- Spectral density is a real, non-negative and even function of frequency (ω or f).
- Since it is an even function of the frequency, a rational spectral density of the form

$$S_X(\omega) = \frac{S_0(\omega^{2n} + a_{2n-2}\omega^{2n-2} + \cdots + a_2\omega^2 + a_0)}{\omega^{2m} + b_{2m-2}\omega^{2m-2} + \cdots + b_2\omega^2 + b_0}$$

  contains only even powers of ω.
- White noise is a process whose spectral density is a constant over the entire frequency range, i.e.,

$$S_X(\omega) = S_0 \quad \text{for all } \omega$$

  (just like a "white" light contains all colored lights with equal intensity). When the marginal distribution is Gaussian, we call it a **Gaussian white noise**.

Spectral Density of Constant or Periodic Signals

Consider a process $X(t) = A + B\cos(2\pi f_0 t + \Theta)$

where A, B and f_0 are constant and Θ is a random variable uniformly distributed over $(0,2\pi)$.

Let $X_T(t)$ be a truncated version of $X(t)$ over $(-T,T)$.

$$F_{X_T}(f) = \int_{-T}^{T}[A + B\cos(2\pi f_0 + \Theta)]e^{-j2\pi ft}dt$$

$$S_X(\omega) = \lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T}$$

$$\mathbf{F}[X(t)] = \int_{-\infty}^{\infty}[A + B\cos(2\pi f_0 + \Theta)]e^{-j2\pi ft}dt$$

$$= A\delta(f) + (B/2)\left[\delta(f + f_0)e^{-j\Theta} + \delta(f - f_0)e^{j\Theta}\right]$$

$$F_{X_T}(f) = 2T\mathrm{sinc}(2Tf) * \mathbf{F}[X(t)]$$

$$= 2AT\mathrm{sinc}(2Tf) + BT\{\mathrm{sinc}[2(f + f_0)T]e^{-j\Theta} + \mathrm{sinc}[2(f - f_0)T]e^{j\Theta}\}$$

$$|F_{X_T}(f)|^2 = F_{X_T}(f)F^*_{X_T}(f)$$

$$= (2AT)^2\mathrm{sinc}^2(2Tf) + (BT)^2\left\{\mathrm{sinc}^2[2(f + f_0)T] + \mathrm{sinc}^2[2(f - f_0)T]\right\}$$

$$+ C(f)e^{-j\Theta} + C(-f)e^{j\Theta} + D(f)e^{-j2\Theta} + D(-f)e^{j2\Theta}$$

Spectral Density of Constant or Periodic Signals

$$E\left[|F_{X_T}(f)|^2\right]$$
$$= (2AT)^2 \mathrm{sinc}^2(2Tf) + (BT)^2\left\{\mathrm{sinc}^2[2(f+f_0)T] + \mathrm{sinc}^2[2(f-f_0)T]\right\}$$

because $E_\Theta[G(f)e^{-j\Theta}] = 0$ and $E_\Theta[H(f)e^{-j2\Theta}] = 0$ for Θ uniformly distributed in $(0,2\pi)$

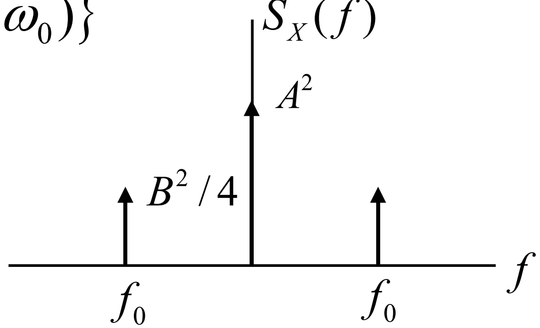
$$S_X(f) = \lim_{T\to\infty} \frac{E\left[|F_{X_T}(f)|^2\right]}{2T}$$
$$= \lim_{T\to\infty}\left\{A^2 2T\mathrm{sinc}^2(2Tf) + (B^2/4)\left(2T\mathrm{sinc}^2[2(f+f_0)T] + 2T\mathrm{sinc}^2[2(f-f_0)T]\right)\right\}$$

But, $\lim_{T\to\infty} 2T\mathrm{sinc}^2(2Tf) = \delta(f)$

Therefore, $S_X(f) = A^2\delta(f) + (B^2/4)\left\{\delta(f+f_0) + \delta(f-f_0)\right\}$

And, $S_X(\omega) = 2\pi A^2\delta(\omega) + (\pi B^2/2)\left\{\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right\}$

The spectral density thus consists of three spikes (delta function) at DC (with height A^2) and at $\pm f_0$ (with height $B^2/4$), respectively.



Mean-Square Value and Total Power

$$X(t) = A + B\cos(\omega_0 t + \Theta)$$ Θ uniformly distributed in $(0,2\pi)$

$$S_X(\omega) = 2\pi A^2\delta(\omega) + (\pi B^2/2)\left\{\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right\}$$

Total power:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{2\pi A^2\delta(\omega) + (\pi B^2/2)\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]\right\}d\omega$$

$$= A^2 + \frac{B^2}{4} + \frac{B^2}{4} = A^2 + \frac{B^2}{2}$$

Mean-Square Value of the process

$$E_\Theta[X^2(t)] = E_\Theta\{[A + B\cos(\omega_0 t + \Theta)]^2\} = E_\Theta\{A^2 + 2AB\cos(\omega_0 t + \Theta) + B^2\cos^2(\omega_0 t + \Theta)\}$$

$$= A^2 + E_\Theta\{2AB\cos(\omega_0 t + \Theta)\} + B^2 E_\Theta[\cos^2(\omega_0 t + \Theta)]$$

$$E_\Theta\{2AB\cos(\omega_0 t + \Theta)\} = 0$$ for $\Theta \sim U(0,2\pi)$

$$E_\Theta\{\cos^2(\omega_0 t + \Theta)\} = E_\Theta\left\{\frac{\cos(2\omega_0 t + 2\Theta) + 1}{2}\right\} = \frac{1}{2}$$

Therefore, $$E_\Theta[X^2(t)] = \overline{X^2} = A^2 + \frac{B^2}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega$$

Example 7-3.1

A stationary random process has a spectral density of the form

$$S_X(f) = 4\delta(f) + 18\delta(f+8) + 18\delta(f-8)$$

a. Find the discrete frequencies present;
b. Find the mean value of the process;
c. Find the variance of the process.

$$S_X(f) = A^2\delta(f) + (B^2/4)\{\delta(f+f_0) + \delta(f-f_0)\}$$

Frequencies present are 0, +8 and –8 Hz respectively.

The mean value of the process corresponds to the amplitude of the DC (I.e., zero-frequency) component which has a power spectral density of 4. Therefore, the mean is ±2.

The mean-square value is 4+18+18=40

Thus, the variance is $\sigma^2 = \overline{X^2} - (\overline{X})^2 = 40 - 4 = 36$

Time Derivative of A Random Process

Let $\dot{X}(t) = dX(t)/dt$

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_X(\omega)e^{j\omega t}d\omega$$

$$\dot{X}(t) = \frac{dX(t)}{dt} = \frac{d}{dt}\frac{1}{2\pi}\int_{-\infty}^{\infty} F_X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F_X(\omega)e^{j\omega t}d\omega$$

$$\mathbf{F}[\dot{X}(t)] = j\omega F_X(\omega)$$

$$S_{\dot{X}}(\omega) = \lim_{T\to\infty}\frac{E[|\, j\omega F_{X_T}(\omega)(-j\omega)F_{X_T}(-\omega)\,|]}{2T} = \omega^2 \lim_{T\to\infty}\frac{E[|\, F_{X_T}(\omega)\,|^2]}{2T} = \omega^2 S_X(\omega)$$

In case $S_X(\omega)$ does not drop off more rapidly than $1/\omega^2$ as $\omega \to \infty$, then the mean-square value of the derivative process will become infinite. The process is then said to be non-differentiable.