ECE 3075A
Random Signals

Lecture 2
Event Counting and Review of Combinatorics

School of Electrical and Computer Engineering
Georgia Institute of Technology
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Syllabus

- Lecturer: Professor B.H. Juang
- Lecture time: MWF 1600-1710 @ Van Leer C240
- Office hours: MF 1500-1555 @ Bunger Henry 310
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  Office hours: Tu: 5 - 6 pm, Th: 5 - 7 pm
  Tutorial Lab: VL C448
- Text: Cooper & McGillem, *Probabilistic Methods of Signal & System Analysis*, 3rd edition
- Homework, lecture note and other information:
  http://users.ece.gatech.edu/~juang

Axiomatic Approach

- Built upon set theory
- A **Probability Space** consists of three components: the observation space, S, the probability measure, Pr(), and the assignment of probability, Pr(event), that satisfies a set of axioms.
- **Observation Space** is a space whose elements are all the outcomes of an experiment.
- **Events** are subsets of the observation space. (σ algebra)
- **Three axioms** to satisfy:

Non-negativity:	$\Pr(A) \ge 0$
Sure event & total probability:	$\Pr(S) = 1$
Exclusivity or additivity:	If $A \cap B = \phi$, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$

Corollaries from the Axioms

$$S \cap \phi = \phi \text{ and } S \cup \phi = S \;\Rightarrow\; \Pr(S \cup \phi) = \Pr(S) = \Pr(S) + \Pr(\phi) \;\Rightarrow\; \Pr(\phi) = 0$$

$$A \cap \overline{A} = \phi \text{ and } A \cup \overline{A} = S \;\Rightarrow\; \Pr(S) = \Pr(A \cup \overline{A}) = \Pr(A) + \Pr(\overline{A}) = 1$$
$$\Rightarrow\; \Pr(A) = 1 - \Pr(\overline{A}) \le 1$$

$$A \cup B = A \cup (\overline{A} \cap B) \text{ and } A \cap (\overline{A} \cap B) = \phi$$
$$\Rightarrow\; \Pr(A \cup B) = \Pr[A \cup (\overline{A} \cap B)] = \Pr(A) + \Pr(\overline{A} \cap B)$$

$$B = (A \cap B) \cup (\overline{A} \cap B) \text{ and } (A \cap B) \cap (\overline{A} \cap B) = \phi \;\Rightarrow$$
$$\Pr(B) = \Pr(A \cap B) + \Pr(\overline{A} \cap B) \quad \text{or} \quad \Pr(\overline{A} \cap B) = \Pr(B) - \Pr(A \cap B)$$

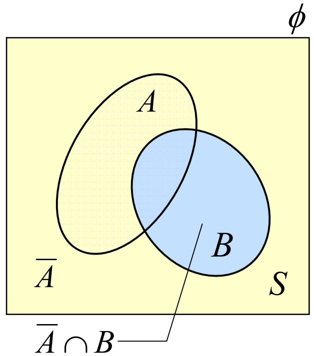
$$\Pr(A \cup B) = \Pr(A) + \Pr(\overline{A} \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \le \Pr(A) + \Pr(B)$$

Important Corollaries

- 0-measure events:

  $\Pr(\phi) = 0$

  Theoretically, there could be other 0-measure events than the empty set – the axioms allow that. But in the relative frequency approach, they are usually either not treated or treated differently as "unseen events".



- Complementary events:

  $\Pr(A) = 1 - \Pr(\overline{A}) \le 1$

- $\Pr(A \cup B) = \Pr(A) + \Pr(\overline{A} \cap B)$
- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \le \Pr(A) + \Pr(B)$

Example

- Die-throwing experiment

  $S = \{1, 2, 3, 4, 5, 6\}$

  Let $\Pr(1) = p_1, \Pr(2) = p_2, \cdots, \Pr(6) = p_6$

  For an unbiased die, we expect $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$

  Let $A = \{1, 4\} = \{1\} \cup \{4\}$, then $\Pr(A) = \Pr(1) + \Pr(4) = p_1 + p_4$

  Let $B = \{1, 5, 6\} = \{1\} \cup \{5\} \cup \{6\}$, then $\Pr(B) = p_1 + p_5 + p_6$

  $C = A \cup B = \{1, 4, 5, 6\} = A \cup (\overline{A} \cap B)$

  $\Pr(C) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

  $= p_1 + p_4 + p_1 + p_5 + p_6 - p_1 = p_1 + p_4 + p_5 + p_6$

Combined Experiments

- The outcomes of experiment 1 form an observation space $S_1 = \{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n\}$ and those of experiment 2 form $S_2 = \{\beta_1, \beta_2, \beta_3, \cdots, \beta_m\}$.
- A **trial** in each experiment would produce an outcome in the respective observation space; the two respective outcomes can be taken together as if produced by a single trial of a **combined experiment.**
- The combined experiment then corresponds to a new observation space $S_1 \times S_2$ in which the elements are 2-tuples $(\alpha_1, \beta_1), (\alpha_1, \beta_2), (\alpha_1, \beta_3), \cdots, (\alpha_n, \beta_m)$
- $S_1 \times S_2$ is called a cartesian product space.
- A combined experiment may involve many experiments.

Example of Combined Experiment

- Consider two experiments: coin-tossing $S_1 = \{\text{H, T}\}$ and die- throwing $S_2 = \{1, 2, 3, 4, 5, 6\}$
- The cartesian product space has 12 elements:

  $S = S_1 \times S_2 = \{(\text{H},1), (\text{H},2), (\text{H},3), (\text{H},4), (\text{H},5), (\text{H},6), (\text{T},1), (\text{T},2), (\text{T},3), (\text{T},4), (\text{T},5), (\text{T},6)\}$
- Examples of events

  $A_1 = \{\text{H}\} \subset S_1, \quad A_2 = \{1, 3, 5\} \subset S_2$

  Then, $A = A_1 \times A_2 = \{(\text{H},1), (\text{H},3), (\text{H},5)\} \subset S_1 \times S_2$
- Probability of combined events of **independent experiments**

  $\Pr(A) = \Pr(A_1 \times A_2) = \Pr(A_1)\Pr(A_2)$

The probability (value) of event A in the cartesian product space is obtained as the product of the probabilities of A_1 and A_2 . But, one should keep in mind that the measures are related to different spaces, respectively.

Revisit to Relative Frequency of Occurrences

- Use the die-throwing experiment as example.
- S has 6 possible outcomes.
- Let $N_i, i = 1, 2, \cdots, 6$, be the number of occurrences of these possible outcomes after $N = \sum_{i=1}^{6} N_i$ trials.
- The relative frequency of occurrences are defined as

$$r_i = N_i / N, \quad i = 1, 2, \cdots, 6$$

- The relative frequency of occurrences can be used as an **empirical estimate** of probability, i.e., $p_i \cong r_i$
- If $N \to \infty$ and the die is unbiased or fair, we expect

$$r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = \frac{1}{6}$$

Events of Equally Probable Outcomes

- Therefore, **if the outcomes are equally probable**, they can serve as the counting unit when calculating the probability of an event.

$$A = \{\text{odd number}\} = \{1, 3, 5\}, \quad \Pr(A) = 3 \times \left(\frac{1}{6}\right) = \frac{1}{2}$$

This event accounts for three out of the six possible outcomes, and thus has probability one half.

$$B = \{\text{number less than } 3\} = \{1, 2\}, \quad \Pr(A) = 2 \times \left(\frac{1}{6}\right) = \frac{1}{3}$$

This event accounts for two out of the six possible outcomes, and thus has probability one third.

This is true only if the elementary outcomes are equally probable.

Bernoulli Trials

- Trials of the same experiment with outcomes combined to form a cartesian product space as in combined experiment. $S \Rightarrow S^n = S \times S \times \cdots \times S$
- Consider an event A.

$$\Pr(A) = p \text{ and } \Pr(\overline{A}) = q = 1 - p$$

- If we repeat the trial n times, assuming the trials are independent, then the outcome (a sequence), say,

$$(A, \overline{A}, \overline{A}, \cdots, A, \overline{A}) \in S^n$$

has probability

$$\Pr[(A, \overline{A}, \overline{A}, \cdots, A, \overline{A})] = \Pr(A)\Pr(\overline{A})\Pr(\overline{A})\cdots\Pr(A)\Pr(\overline{A}) = p^k q^{n-k}$$

assuming there are k and n-k occurrences of A and $\overline{A}$ in $(A, \overline{A}, \overline{A}, \cdots, A, \overline{A})$, respectively.

Cardinality of A Cartesian Space

- Let $\| S \| = (\text{number of elements in } S) = (\text{cardinality of } S) = I$
- Then $\| S^n \| = I^n$
- If all the elements in S^n are equiprobable, then

$$\Pr(B) = \frac{1}{I^n} \quad \text{for all } B \in S^n$$

Example:

Assume $p = q = 0.5$. Then,

$$\Pr(B) = \Pr[(A, \overline{A}, \overline{A}, \cdots, A, \overline{A})] = p^k q^{n-k} = \left(\frac{1}{2}\right)^n$$

If $p \neq q$, then the outcome sequences in S^n are not equiprobable.

Combinatorics

- Consider binary sequences of length n in which event A occurs exactly k times and $\overline{A}$ exactly n-k times.
- Among the entire 2^n possible sequences, $\frac{n!}{k!(n-k)!}$ of them are of this kind.
- Binomial coefficient $_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ $\quad (p+q)^n = ?!$

Let event B_k={event A occurs k times in n trials}

$$\Pr(B_k) \equiv p_n(k) = \binom{n}{k} p^k q^{n-k}$$

DeMoivre-Laplace Theorem

$npq >> 1$ and

$|k\text{-}np| <\sim \sqrt{npq}$

$$p_n(k) = \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)^2/2npq}$$

Interpretation of DeMoivre-Laplace Theorem

- When the number of trials is large, i.e., $npq >> 1$ the mean approaches np and the variance approaches $np(1\text{-}p)=npq$
- Law of large numbers – asymptotic normality
- Deviation from normal distribution is small if k is in the vicinity of one standard deviation, i.e. $|k\text{-}np| <\sim \sqrt{npq} = \sigma$

$npq >> 1$ and $|k\text{-}np| <\sim \sqrt{npq}$

$$p_n(k) = \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)^2/2npq}$$