## ECE 3075A
## *Random Signals*

### Lecture 20
# Spectral Density And Autocorrelation Function

School of Electrical and Computer Engineering
Georgia Institute of Technology
Summer, 2003



## Spectral Density & Autocorrelation Function

$$F_{X_T}(\omega) = \int_{-\infty}^{\infty} X_T(t)e^{-j\omega t}dt \qquad F^*_{X_T}(\omega) = \int_{-\infty}^{\infty} X_T(t)e^{j\omega t}dt = F_{X_T}(-\omega)$$

$$S_X(\omega) = \lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T} = \lim_{T\to\infty}\frac{1}{2T}E\left\{\int_{-\infty}^{\infty} X_T(t_1)e^{+j\omega t_1}dt_1\int_{-\infty}^{\infty} X_T(t_2)e^{-j\omega t_2}dt_2\right\}$$

$$S_X(\omega) = \lim_{T\to\infty}\frac{1}{2T}E\left\{\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{\infty} X_T(t_1)X_T(t_2)e^{+j\omega(t_2-t_1)}dt_2\right\}$$

$$= \lim_{T\to\infty}\frac{1}{2T}\left\{\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{\infty} E[X_T(t_1)X_T(t_2)]e^{+j\omega(t_2-t_1)}dt_2\right\}$$

$$E[X_T(t_1)X_T(t_2)] = \begin{cases} R_X(t_1,t_2), & |t_1|,|t_2| \le T \\ 0, & \text{elsewhere}\end{cases}$$

$$S_X(\omega) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T-t_1}^{T-t_1} d\tau\int_{-T}^{T} R_X(t_1,t_1+\tau)e^{-j\omega\tau}dt_1$$

$$= \int_{-\infty}^{\infty}\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_X(t_1,t_1+\tau)dt_1\right\}e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty}\langle R_X(t_1,t_1+\tau)\rangle e^{-j\omega\tau}d\tau$$



## Spectral Density of A Stationary Process

If $X(t)$ is a stationary ergodic process $\langle R_X(t_1,t_1+\tau)\rangle = R_X(\tau)$

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau$$

The spectral density is the Fourier transform of the autocorrelation function. The autocorrelation function is the inverse Fourier transform of the spectral density function.

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega\tau}d\omega$$

Since the autocorrelation function is a real, even function,

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)(\cos\omega\tau - j\sin\omega\tau)d\tau = 2\int_0^{\infty} R_X(\tau)\cos\omega\tau d\tau$$

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)(\cos\omega\tau + j\sin\omega\tau)d\omega = \frac{1}{\pi}\int_0^{\infty} S_X(\omega)\cos\omega\tau d\omega$$

## Example – Exponential Autocorrelation

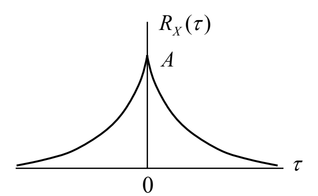
$$R_X(\tau) = Ae^{-\beta|\tau|} \qquad A > 0,\ \beta > 0$$

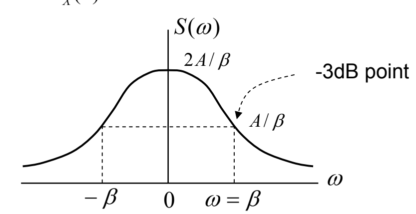
$$S_X(\omega) = \int_{-\infty}^{0} Ae^{\beta\tau}e^{-j\omega\tau}d\tau + \int_0^{\infty} Ae^{-\beta\tau}e^{-j\omega\tau}d\tau = A\frac{e^{(\beta-j\omega)\tau}}{\beta-j\omega}\Bigg|_{-\infty}^{0} + A\frac{e^{-(\beta+j\omega)\tau}}{-(\beta+j\omega)}\Bigg|_{0}^{\infty}$$

$$= A\left[\frac{1}{\beta-j\omega} + \frac{1}{\beta+j\omega}\right] = \frac{2\beta A}{\beta^2+\omega^2}$$

$$S_X(\omega=0) = \frac{2\beta A}{\beta^2+0^2} = \frac{2A}{\beta} \qquad S_X(\omega=\beta) = \frac{2\beta A}{\beta^2+\beta^2} = \frac{A}{\beta} = \frac{1}{2}S_X(0)$$

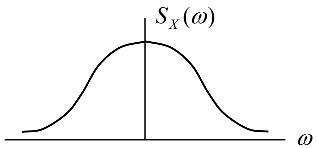
$$\frac{S_X(\beta)}{S_X(0)} = \frac{1}{2}$$ Or in dB: $$10\log_{10}\frac{S_X(\beta)}{S_X(0)} = 10\log_{10}\frac{1}{2} = -3.0103\text{ dB}$$



# Bandwidth of Spectral Density

When $X(t)$ has a spectral density with components clustered around $\omega = 0$, it is called a lowpass process (signal) or a baseband process (signal).




Given $S_X(\omega)$, which is non-negative and real, one can (heuristically) treat

$$p(\omega) = \frac{S_X(\omega)}{\int_{-\infty}^{\infty} S_X(\omega)d\omega} \qquad \text{note}: \int_{-\infty}^{\infty} p(\omega)d\omega = 1$$

as a function similar to a probability density function.

Then, we can measure the spread of the power density by calculating its second moment (mean-square value):

$$B_{rms}^2 = \int_{-\infty}^{\infty} \omega^2 p(\omega)d\omega = \frac{\int_{-\infty}^{\infty} \omega^2 S_X(\omega)d\omega}{\int_{-\infty}^{\infty} S_X(\omega)d\omega}$$

$B_{rms}$ is called the rms bandwidth (equivalent to std, around mean 0)



# Example - Bandwidth

Given the spectral density $S_X(\omega) = \frac{1}{[1+(\omega/10)^2]^2}$

- Find the rms bandwidth;
- Find the frequency at which the spectral density is maximum;
- Find the frequency at which the spectral density is -6 dB from its maximum.

$$\int_{-\infty}^{\infty} S_X(\omega)d\omega = \int_{-\infty}^{\infty} \frac{1}{[1+(\omega/10)^2]^2}d\omega = \int_{-\infty}^{\infty} \frac{10^4}{[10^2+\omega^2]^2}d\omega = 10^4\left\{\frac{\omega}{2\times 10^2(10^2+\omega^2)} + \frac{1}{2\times 10^3}\tan^{-1}\left(\frac{\omega}{10}\right)\right\}\Bigg|_{-\infty}^{\infty} = 5\pi$$

$$B_{rms}^2 = \frac{1}{5\pi}\int_{-\infty}^{\infty} \frac{\omega^2}{[1+(\omega/10)^2]^2}d\omega = \frac{500\pi}{5\pi} = 100, \quad \text{therefore } B_{rms} = 10$$

$S_X(\omega)$ as given is a monotonically decreasing funtion of $\omega$.

Its maximum occurs at $\omega = 0$ and $S_X(\omega = 0) = 1$.

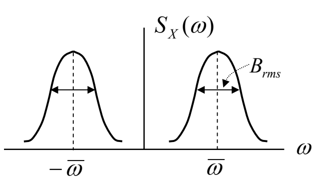
$$-6 = 10\log_{10} S_X(\omega) - 10\log_{10} S_X(0) = 10\log_{10} S_X(\omega)$$

$$S_X(\omega) = \frac{1}{10^{0.6}} = \frac{1}{4} = \frac{1}{[1+(\omega/10)^2]^2}, \Rightarrow 1+(\omega/10)^2 = 2 \Rightarrow \omega/10 = 1 \Rightarrow \omega = 10 \ \text{(rad/s)}$$



# Bandwidth of Spectral Density

When $X(t)$ has a spectral density with components clustered around $\omega = \pm\overline{\omega}$, it is called a bandpass process (signal).




The center of the spectral density can be defined as the mean frequency:

$$\overline{\omega} = \int_{-\infty}^{\infty} \omega p(\omega)d\omega = \frac{\int_0^{\infty} \omega S_X(\omega)d\omega}{\int_0^{\infty} S_X(\omega)d\omega}$$

Correspondingly, the rms bandwidth is now

$$B_{rms} = 2\left\{\int_0^{\infty} (\omega-\overline{\omega})^2 p_+(\omega)d\omega\right\}^{1/2} = 2\left\{\frac{\int_0^{\infty} (\omega-\overline{\omega})^2 S_X(\omega)d\omega}{\int_0^{\infty} S_X(\omega)d\omega}\right\}^{1/2}$$

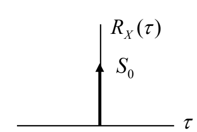
where $p_+(\omega)$ is the positive frequency side of $S_X(\omega)$.

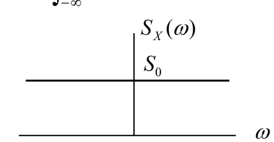
# More on White Noise

White noise is a process whose spectral density is a constant for all frequencies: $S_X(\omega) = S_0$

The corresponding autocorrelation function is a delta-function at 0 lag ($\tau = 0$): $R_X(\tau) = S_0\delta(\tau)$

Note: $S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} S_0\delta(\tau)e^{-j\omega\tau}d\tau = S_0$




Particular care for the concept of white noise:

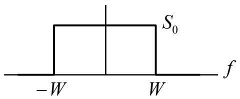
In theory: $\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0 d\omega = \infty$

Not a realizable process!!

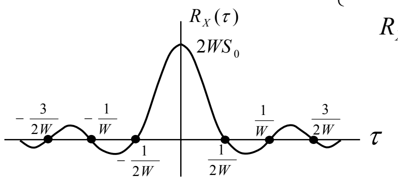
# Autocorrelation of Bandlimited White Noise

A more useful concept is the bandlimited white noise whose spectral density is a constant over a finite bandwidth and zero outside the frequency range. For example:

$$S_X(\omega) = \begin{cases} S_0, & |\omega| \le 2\pi W \\ 0, & |\omega| > 2\pi W \end{cases}$$




$$R_X(\tau) = \mathbf{F}^{-1}\{S_X(f)\} = \mathbf{F}^{-1}\left\{S_0 \mathrm{rect}\left(\frac{f}{2W}\right)\right\} = 2WS_0 \mathrm{sinc}(2W\tau)$$




$R_X(\tau) = 0$ at $\tau = n/2W$, $n = \pm 1, \pm 2, \cdots$

Random variables from a band-limited white noise are uncorrelated if they are separated in time by any multiple of $1/2W$ seconds.

Therefore, if a continuous time bandlimited white noise process is sampled at twice the maximum frequency limit ($2W$), then the resultant samples of the discrete time sequence are uncorrelated.



# Cross-Spectral Density

- Just as we are interested in cross-correlation analysis (e.g., to investigate the joint statistical behavior of the input and the output of a system), we are interested in cross-spectral density, which is the frequency domain representation of the cross-correlation function.

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E[F_{X_T}(-\omega)F_{Y_T}(\omega)]}{2T} \quad \text{and} \quad S_{YX}(\omega) = \lim_{T\to\infty} \frac{E[F_{Y_T}(-\omega)F_{X_T}(\omega)]}{2T}$$

Key properties:

1. $S_{XY}(\omega) = S^*_{YX}(\omega)$ (* denotes complex conjugate)
2. $\mathrm{Re}\{S_{XY}(\omega)\}$ is an even function of $\omega$. Also true for $S_{YX}(\omega)$ .
3. $\mathrm{Im}\{S_{YX}(\omega)\}$ is an odd function of $\omega$. Also true for $S_{YX}(\omega)$ .



# Cross-spectral Density and Cross-correlation

- Similar to the relationship between autocorrelation function and power spectral density,

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau \qquad S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega \qquad R_{YX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega) e^{j\omega\tau} d\omega$$

Example:

For two jointly stationary random processes, the crosscorrelation function is

$$R_{XY}(\tau) = \begin{cases} 2e^{-2\tau}, & \tau > 0 \\ 0, & \tau < 0 \end{cases}$$

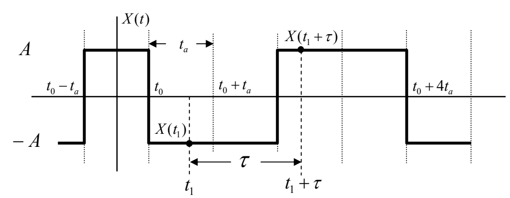
The corresponding cross-spectral density is

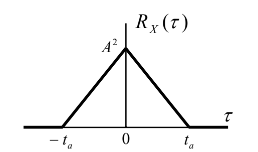
$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau = \int_0^{\infty} 2e^{-(j\omega+2)\tau} d\tau = \frac{2\exp[-(j\omega+2)\tau]}{-(j\omega+2)}\Big|_0^{\infty} = \frac{2}{j\omega+2}$$

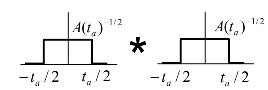
$$S_{YX}(\omega) = S^*_{XY}(\omega) = \frac{2}{-j\omega+2}$$

# Examples and Applications

- Binary Process




$A(t_a)^{-1/2} \bullet \mathrm{rect}(t/t_a)$

$$R_X(\tau) = \begin{cases} A^2\left[\frac{t_a - |\tau|}{t_a}\right] = A^2\left[1 - \frac{|\tau|}{t_a}\right], & |\tau| \le t_a \\ 0, & |\tau| > t_a \end{cases}$$

$$A(t_a)^{-1/2} \bullet \mathrm{rect}(t/t_a) \overset{\mathbf{F}}{\Leftrightarrow} A(t_a)^{1/2}\mathrm{sinc}(t_a\omega/2\pi)$$

$$S_X(\omega) = 2\int_0^{t_a} A^2\left(1 - \frac{\tau}{t_a}\right) e^{-j\omega\tau} d\tau$$

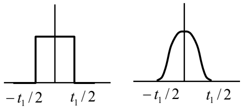
$$= \mathbf{F}[A \bullet \mathrm{Rect}(t/t_a)] \bullet \mathbf{F}[A \bullet \mathrm{Rect}(t/t_a)] = A^2 t_a \mathrm{sinc}^2(t_a\omega/2\pi)$$

# Spectral Density of Pulses

Consider the following elementary pulses that are often used:

$p_r(t) = \text{rect}(t/t_1)$ and

$$p_c(t) = \frac{1}{2}\left(1+\cos\frac{2\pi t}{t_1}\right), \quad |t| \le \frac{t_1}{2}; \;= 0, \quad |t| > \frac{t_1}{2}.$$




A process formed with these elementary pulses can be defined:

$$X(t) = \sum_{i=-\infty}^{\infty} G(i) A p(t - it_1)$$

where $G(i)$ is i.i.d. with $\Pr\{G(i)=-1\} = \Pr\{G(i)=1\} = 0.5$, $p$ is either $p_r$ or $p_c$, and $A$ is a constant that defines the pulse amplitude.

$X(t)$ is thus sum of many independent, zero-mean processes because $E\{G(i)\} = 0$. Using previous results, we have

$$R_{X_T}(t, t+\tau) = A^2 \sum_{i=-I}^{I} R_P(t - it_1, t - it_1 + \tau), \quad |\tau| \le t_1/2, \text{ where } I = T/t_1,$$

and $R_p(t, t+\tau)$ is the autocorrelation of the elementary pulse.





# Spectral Density of Pulses

Or more formally, $X_T(t) = \sum_{i=-I}^{I} G(i) A p(t - it_1)$

$$E[X_T(t) X_T(t')] = E\left[\sum_{i=-I}^{I} G(i) A p(t - it_1) \sum_{j=-I}^{I} G(j) A p(t' - jt_1)\right]$$

$$= \sum_{i=-I}^{I}\sum_{j=-I}^{I} E[G(i)G(j)] A p(t - it_1) A p(t' - jt_1)$$

$$= \sum_{i=-I}^{I}\sum_{j=-I}^{I} \delta(i-j) A p(t - it_1) A p(t' - jt_1) = \sum_{i=-I}^{I} A^2 p(t - it_1) p(t' - it_1)$$

$$E[|F_{X_T}(\omega)|^2] = E\left\{\left[\int_{-\infty}^{\infty} X_T(t) e^{-j\omega t} dt\right]\left[\int_{-\infty}^{\infty} X_T(t') e^{j\omega t'} dt'\right]\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X_T(t) X_T(t')] e^{-j\omega t} e^{j\omega t'} dt dt' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{i=-I}^{I} A^2 p(t - it_1) p(t' - it_1) e^{-j\omega t} e^{j\omega t'} dt dt'$$

$$= \sum_{i=-I}^{I} A^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(t) p(t') e^{-j\omega(t+it_1)} e^{j\omega(t'+it_1)} dt dt' = \sum_{i=-I}^{I} A^2 |F_P(\omega)|^2 = 2IA^2 |F_P(\omega)|^2$$

$$= \frac{2T}{t_1} A^2 |F_P(\omega)|^2 \quad \Rightarrow \quad S_X(\omega) = \lim_{T\to\infty} \frac{E[|F_{X_T}(\omega)|^2]}{2T} = \frac{A^2 |F_P(\omega)|^2}{t_1}$$