ECE 3075A

Random Signals

Lecture 21

Spectral Density And Introduction to System Analysis

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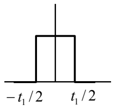
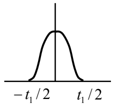
Spectral Density of Pulses

Consider the following elementary pulses that are often used:

$p_r(t) = \mathrm{rect}(t/t_1)$ and

$p_c(t) = \frac{1}{2}\left(1+\cos\frac{2\pi t}{t_1}\right), \quad |t| \le \frac{t_1}{2}; \ = 0, \quad |t| > \frac{t_1}{2}.$





A process formed with these elementary pulses can be defined:

$$X(t) = \sum_{i=-\infty}^{\infty} G(i)Ap(t - it_1)$$

where $G(i)$ is i.i.d. with $\Pr\{G(i) = -1\} = \Pr\{G(i) = 1\} = 0.5$, p is either p_r or p_c, and A is a constant that defines the pulse amplitude.

$X(t)$ is thus sum of many independent, zero-mean processes because $E\{G(i)\} = 0$. Using previous results, we have

$$R_{X_T}(t, t+\tau) = A^2 \sum_{i=-I}^{I} R_P(t - it_1, t - it_1 + \tau), \ |\tau| \le t_1/2, \text{ where } I = T/t_1,$$

and $R_p(t, t+\tau)$ is the autocorrelation of the elementary pulse.

Spectral Density of Pulses

Or more formally, $X_T(t) = \sum_{i=-I}^{I} G(i)Ap(t - it_1)$

$$E[X_T(t)X_T(t')] = E\left[\sum_{i=-I}^{I} G(i)Ap(t-it_1)\sum_{j=-I}^{I} G(j)Ap(t'-jt_1)\right]$$

$$= \sum_{i=-I}^{I}\sum_{j=-I}^{I} E[G(i)G(j)]Ap(t-it_1)Ap(t'-jt_1)$$

$$= \sum_{i=-I}^{I}\sum_{j=-I}^{I} \delta(i-j)Ap(t-it_1)Ap(t'-jt_1) = \sum_{i=-I}^{I} A^2 p(t-it_1)p(t'-it_1)$$

$$E[|F_{X_T}(\omega)|^2] = E\left\{\left[\int_{-\infty}^{\infty} X_T(t)e^{-j\omega t}dt\right]\left[\int_{-\infty}^{\infty} X_T(t')e^{j\omega t'}dt'\right]\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X_T(t)X_T(t')]e^{-j\omega t}e^{j\omega t'}dtdt' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{i=-I}^{I} A^2 p(t-it_1)p(t'-it_1)e^{-j\omega t}e^{j\omega t'}dtdt'$$

$$= \sum_{i=-I}^{I} A^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(t)p(t')e^{-j\omega(t+it_1)}e^{j\omega(t'+it_1)}dtdt' = \sum_{i=-I}^{I} A^2|F_P(\omega)|^2 = 2IA^2|F_P(\omega)|^2$$

$$= \frac{2T}{t_1}A^2|F_P(\omega)|^2 \quad \Rightarrow \quad S_X(\omega) = \lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T} = \frac{A^2|F_P(\omega)|^2}{t_1}$$

Spectral Density of Binary Processes

For $p(t) = p_r(t) = \mathrm{rect}(t/t_1)$,

$F_P(\omega) = t_1\mathrm{sinc}(t_1\omega/2\pi)$ and thus $|F_P(\omega)|^2 = t_1^2\mathrm{sinc}^2(t_1\omega/2\pi)$

Therefore, $$S_X(\omega) = \frac{A^2|F_P(\omega)|^2}{t_1} = A^2 t_1 \mathrm{sinc}^2(t_1\omega/2\pi)$$

For $p(t) = p_c(t) = \frac{1}{2}\left(1+\cos\frac{2\pi t}{t_1}\right), \quad |t| \le \frac{t_1}{2}; \ = 0, \quad |t| > \frac{t_1}{2},$

$$F_P(\omega) = \frac{1}{2}\int_{-t_1/2}^{t_1/2}\left(1+\cos\frac{2\pi t}{t_1}\right)e^{-j\omega t}dt = \frac{t_1}{2}\left[\frac{\sin(\omega t_1/2)}{(\omega t_1/2)}\right]\left[\frac{\pi^2}{\pi^2 - (\omega t_1/2)^2}\right]$$

$$S_X(\omega) = \frac{A^2 t_1}{4}\left[\frac{\sin(\omega t_1/2)}{(\omega t_1/2)}\right]^2\left[\frac{\pi^2}{\pi^2 - (\omega t_1/2)^2}\right]^2$$

$$S_X(f) = \frac{A^2 t_1}{4}\mathrm{sinc}^2(t_1 f)\left[\frac{1}{1-(t_1 f)^2}\right]^2$$

Note that in both cases, max $S_X(\omega)$ occurs at $\omega = 0$

Example

The spectral density of the pulses thus defines the bandwidth of the binary signal carried by these pulses. This bandwidth is a function of the pulse width t_1.

We often need to "match" the bandwidth of the signal to the bandwidth of the transmission channel so as to reduce undesirable distortion of interference.

For example, we may request a bandwidth which would support transmission of the signal up to the frequency at which the spectral density is no more than 1% of its maximum. What would this bandwidth be?

$$\frac{S_X(f)}{S_X(0)} \le 0.01 \quad \text{for } |f| > f_1$$

For the rectangular pulse,

$$S_X(0) = A^2 t_1 \qquad S_X(f_1) = A^2 t_1 \text{sinc}^2(t_1 f_1) = 0.01 A^2 t_1 \Rightarrow \text{sinc}^2(t_1 f_1) = 0.01$$

$$t_1 f_1 \pi = 8.4226 \quad \Rightarrow \quad f_1 = 2.681 / t_1$$

For the raised cosine pulse,

$$S_X(0) = A^2 t_1 / 4$$

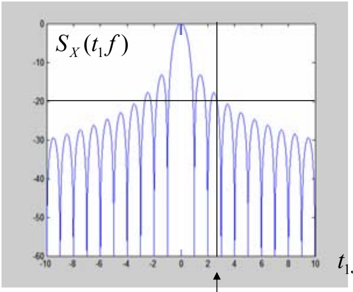
$$S_X(f_1) = \frac{A^2 t_1}{4} \text{sinc}^2(t_1 f_1) \left[\frac{1}{1-(t_1 f_1)^2} \right]^2 = 0.01 \frac{A^2 t_1}{4}$$

$$t_1 f_1 \pi = 5.1836 \quad \Rightarrow \quad f_1 = 1.65 / t_1$$

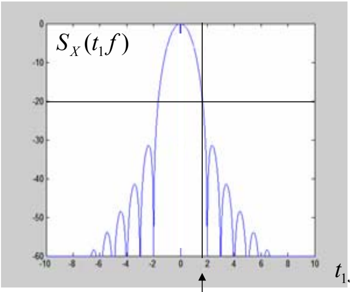
Bandwidth of Various Pulses

$$\frac{S_X(f)}{S_X(0)} \le 0.01 \quad \text{for } |f| > f_1 \quad \Rightarrow \quad 10 \log_{10}(0.01) = -20 \text{ (dB) at } f_1$$

Rectangular pulse



Raised-cosine pulse



Example

An nth- order Butterworth spectrum is one whose spectral density is given by

$$S_X(f) = \frac{1}{1+(f/W)^{2n}}$$

in which W is the so-called half-power bandwidth.

1. Find the bandwidth outside of which the spectral density is less than 1% of its maximum value.
2. For n=1, find the bandwidth (F) outside of which no more than 1% of the average power exists.

$$\max S_X = S_X(0) = \frac{1}{1+(0/W)^{2n}} = 1$$

$S_X(f)$ is a monotonically decreasing function of f.

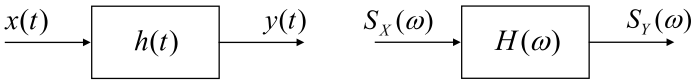
$$S_X(f) = \frac{1}{1+(f/W)^{2n}} = 0.01 \quad \Rightarrow \quad 100 = 1 + (f/W)^{2n} \quad \Rightarrow \quad f = W(99)^{1/(2n)}$$

$$\int_{-\infty}^{\infty} S_X(f)df = \int_{-\infty}^{\infty} \frac{1}{1+(f/W)^2} df = W \tan^{-1}\left(\frac{f}{W}\right)\Bigg|_{-\infty}^{\infty} = W\pi$$

$$\int_p^q \frac{2a}{(2\pi f)^2 + a^2} df = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{2\pi f}{a}\right)\right]_p^q$$

$$\int_{-F}^{F} \frac{1}{1+(f/W)^2} df = W \tan^{-1}\left(\frac{f}{W}\right)\Bigg|_{-F}^{F} = W 2 \tan^{-1}\left(\frac{F}{W}\right) = 0.99 W\pi \qquad F = 63.657W$$

System Analysis



Time-domain analysis Frequency-domain analysis

- Only interested in bounded-input/bounded-output systems.
- The output of the system, $y(t)$, is the result of convolution between the input (excitation to the system), $x(t)$, and the **impulse response**, $h(t)$, of the system.

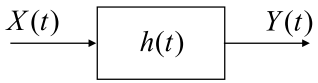
$$y(t) = \int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)d\lambda$$

- For the system to be realizable and stable,

$$h(t) = 0 \text{ for } t < 0 \quad \text{(causality) and} \quad \int_{-\infty}^{\infty} |h(t)| dt < \infty \quad \text{(stability)}$$

Thus, $$y(t) = \int_0^{\infty} x(t-\lambda)h(\lambda)d\lambda = \int_{-\infty}^{0} x(\lambda)h(t-\lambda)d\lambda$$

Random Input to A System



$x(t) \Rightarrow X(t) \qquad y(t) \Rightarrow Y(t)$

The input is no longer a fixed function.

The same linear system concept applies.

Example: $h(t) = \begin{cases} 5e^{-3t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$X(t) = M + 4\cos(2t + \Theta)$ where M is a random variable and Θ is an independent random variable, uniformly distributed in $(0,2\pi)$.

$$Y(t) = \int_{-\infty}^{t} [M + 4\cos(2\lambda + \Theta)] 5e^{-3(t-\lambda)} d\lambda$$

$$= \frac{5}{3}M + \frac{20}{13}[3\cos(2t+\Theta) + 2\sin(2t+\Theta)]$$

$Y(t)$ is also a random process whose statistical properties can be derived from the distributions of the random variables, M and Θ.

Example

A linear system has an impulse response of the form

$$h(t) = \begin{cases} te^{-2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and an input that is a random process of the form

$$X(t) = M, \quad -\infty < t < \infty$$

where M is a random variable uniformly distributed in (0,12).

1. Write an expression for the output;
2. Find the mean value of the output;
3. Find the variance of the output.

$$Y(t) = \int_0^{\infty} M\lambda e^{-2\lambda} d\lambda = M \frac{e^{-2\lambda}}{4}(-2\lambda - 1)\Big|_0^{\infty} = \frac{M}{4}$$

$$E[Y(t)] = E[M/4] = 6/4 = 3/2$$

$$E[Y^2(t)] = E[M^2/16] = (144/3)/16 = 3$$

$$\sigma_Y^2 = E[Y^2(t)] - \{E[Y(t)]\}^2 = 3 - (9/4) = 3/4$$

Example

A linear system has an impulse response of the form

$$h(t) = \begin{cases} 5\delta(t) + 3, & 0 \leq t < 1 \\ 0, & \text{elsewhere} \end{cases}$$

The input process is of the form $X(t) = 2\cos(2\pi t + \Theta), \quad -\infty < t < \infty$ where Θ is a random variable uniformly distributed in $(0,2\pi)$.

1. Write an expression for the output process;
2. Find the mean value of the output;
3. Find the variance of the output.

$$Y(t) = \int_0^1 2\cos[2\pi(t-\lambda) + \Theta][5\delta(\lambda) + 3] d\lambda$$

$$= 10\cos(2\pi t + \Theta) - \frac{6}{2\pi}\sin[2\pi(t-\lambda) + \Theta]\Big|_{\lambda=0}^{1} = 10\cos(2\pi t + \Theta)$$

$$E[Y(t)] = E_\Theta[10\cos(2\pi t + \Theta)] = 0$$

$$E[Y^2(t)] = E_\Theta[100\cos^2(2\pi t + \Theta)] = E_\Theta[50 + 50\cos(4\pi t + 2\Theta)] = 50$$

$$\sigma_Y^2 = E[Y^2(t)] - \{E[Y(t)]\}^2 = 50 - 0 = 50$$

Mean of System Output

As demonstrated in the previous examples,

$$\overline{Y} = E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(t-\lambda)h(\lambda)d\lambda\right]$$

In general, for $E\left[\int_{t_1}^{t_2} Z(t)f(t)dt\right] = \int_{t_1}^{t_2} E[Z(t)]f(t)dt$, it requires that

1. $\int_{t_1}^{t_2} E[|Z(t)|]\,|f(t)|\,dt$
2. $Z(t)$ is bounded on the interval (t_1, t_2).

In most cases, we assume these conditions are satisfied.

And if $X(t)$ is wide sense stationary with $E[X(t)] = \overline{X}$, then

$$\overline{Y} = \int_{-\infty}^{\infty} E[X(t-\lambda)]h(\lambda)d\lambda = \overline{X}\int_{-\infty}^{\infty} h(\lambda)d\lambda$$

Note that $\int_{-\infty}^{\infty} h(\lambda)d\lambda$ is the dc gain of the system; the dc component of the output is thus equal to the dc component of the input times the dc gain of the system.

Mean-Square Value of System Output

$$\overline{Y^2} = E[Y^2(t)] = E\left[\int_0^\infty X(t-\lambda_1)h(\lambda_1)d\lambda_1 \bullet \int_0^\infty X(t-\lambda_2)h(\lambda_2)d\lambda_2\right]$$

$$= E\left[\int_0^\infty d\lambda_1 \int_0^\infty X(t-\lambda_1)X(t-\lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_2\right]$$

$$= \int_0^\infty d\lambda_1 \int_0^\infty E[X(t-\lambda_1)X(t-\lambda_2)]h(\lambda_1)h(\lambda_2)d\lambda_2$$

But, $E[X(t-\lambda_1)X(t-\lambda_2)] = R_X(t-\lambda_1-t+\lambda_2) = R_X(\lambda_2-\lambda_1)$

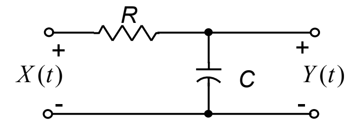
Therefore, $\overline{Y^2} = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2$

Example: For white noise input with $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$, the output "noise" power is

$$\overline{Y^2} = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2$$

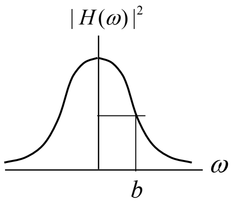
$$= \int_0^\infty d\lambda_1 \int_0^\infty \frac{N_0}{2}\delta(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2 = \frac{N_0}{2}\int_0^\infty h^2(\lambda)d\lambda$$

Analysis of A Simple System



$$h(t) = \begin{cases} be^{-bt}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

where $b = \frac{1}{RC}$



$$\overline{Y} = \overline{X}\int_0^\infty be^{-b\lambda}d\lambda = \overline{X}\left.\frac{be^{-b\lambda}}{-b}\right|_0^\infty = \overline{X}$$

When the input is white noise with $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$

$$\overline{Y^2} = \frac{N_0}{2}\int_0^\infty b^2 e^{-2b\lambda}d\lambda = b^2\frac{N_0}{2}\left.\frac{e^{-2b\lambda}}{-2b}\right|_0^\infty = \frac{bN_0}{4}$$

$$\mathbf{F}\{h(t)\} = H(\omega) = \frac{b}{b+j\omega} \quad \text{(see Table A5);} \quad |H(\omega)|^2 = \frac{b}{b+j\omega}\frac{b}{b-j\omega} = \frac{b^2}{b^2+\omega^2}$$

$|H(\omega=b)|^2 = \frac{b^2}{2b^2} = \frac{1}{2} = \frac{1}{2}|H(\omega=0)|^2$ ⇨ b is the half-power bandwidth of the system

Let $W_{1/2} = \frac{b}{2\pi}$ Hz, then $\overline{Y^2} = \pi W_{1/2}\frac{N_0}{2}$

Example

A linear system has an impulse response $h(t) = te^{-2t}u(t)$ where u(t) is the unit step function. The input to this system is a white noise process having a 2-sided spectral density of $2\ \mathrm{V}^2/\mathrm{Hz}$ plus a dc component of 2 V.

1. Find the mean value of the output of the system;
2. Find the variance of the output;
3. Find the mean-square value of the output.

$X(t) = 2 + V(t)$ with $R_V(\tau) = 2\delta(\tau)$, $\overline{X} = 2 + E[V(t)] = 2$ and $R_X(\tau) = 4 + 2\delta(\tau)$

$$\overline{Y} = \overline{X}\int_0^\infty te^{-2t}dt = 2\left.\frac{e^{-2t}}{4}(-2t-1)\right|_0^\infty = \frac{1}{2}$$

$$\overline{Y^2} = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2 = \int_0^\infty d\lambda_1 \int_0^\infty [4+2\delta(\lambda_2-\lambda_1)]h(\lambda_1)h(\lambda_2)d\lambda_2$$

$$= 4\left[\int_0^\infty te^{-2t}dt\right]^2 + 2\int_0^\infty t^2e^{-4t}dt = 4\left[\left.\frac{e^{-2t}}{4}(-2t-1)\right|_0^\infty\right]^2 + 2\left.\frac{e^{-4t}}{-64}(16t^2+8t+2)\right|_0^\infty$$

$$= \frac{1}{4} + \frac{1}{16} = \frac{5}{16} \qquad \sigma_Y^2 = \overline{Y^2} - \overline{Y}^2 = \frac{5}{16} - \frac{1}{4} = \frac{1}{16}$$