ECE 3075A

Random Signals

Lecture 21

Spectral Density And Introduction to System Analysis

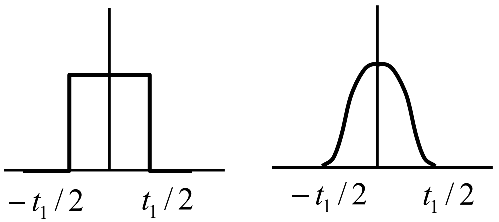
School of Electrical and Computer Engineering
Georgia Institute of Technology
Summer, 2003

Spectral Density of Pulses

Consider the following elementary pulses that are often used:

$p_r(t) = \mathrm{rect}(t/t_1)$ and

$$p_c(t) = \frac{1}{2}\left(1+\cos\frac{2\pi t}{t_1}\right), \quad |t| \le \frac{t_1}{2}; \ = 0, \quad |t| > \frac{t_1}{2}.$$



A process formed with these elementary pulses can be defined:

$$X(t) = \sum_{i=-\infty}^{\infty} G(i) A p(t - it_1)$$

where $G(i)$ is i.i.d. with $\Pr\{G(i)=-1\} = \Pr\{G(i)=1\} = 0.5$, p is either p_r or p_c, and A is a constant that defines the pulse amplitude.

$X(t)$ is thus sum of many independent, zero-mean processes because $E\{G(i)\}=0$. Using previous results, we have

$$R_{X_T}(t, t+\tau) = A^2 \sum_{i=-I}^{I} R_P(t - it_1, t - it_1 + \tau), \ |\tau| \le t_1/2, \text{ where } I = T/t_1,$$

and $R_p(t, t+\tau)$ is the autocorrelation of the elementary pulse.

Spectral Density of Pulses

Or more formally, $X_T(t) = \sum_{i=-I}^{I} G(i) Ap(t - it_1)$

$$E[X_T(t)X_T(t')] = E\left[\sum_{i=-I}^{I} G(i)Ap(t-it_1)\sum_{j=-I}^{I} G(j)Ap(t'-jt_1)\right]$$

$$= \sum_{i=-I}^{I}\sum_{j=-I}^{I} E[G(i)G(j)]Ap(t-it_1)Ap(t'-jt_1)$$

$$= \sum_{i=-I}^{I}\sum_{j=-I}^{I} \delta(i-j)Ap(t-it_1)Ap(t'-jt_1) = \sum_{i=-I}^{I} A^2 p(t-it_1)p(t'-it_1)$$

$$E[|F_{X_T}(\omega)|^2] = E\left\{\left[\int_{-\infty}^{\infty} X_T(t)e^{-j\omega t}dt\right]\left[\int_{-\infty}^{\infty} X_T(t')e^{j\omega t'}dt'\right]\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X_T(t)X_T(t')]e^{-j\omega t}e^{j\omega t'}dtdt' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{i=-I}^{I} A^2 p(t-it_1)p(t'-it_1)e^{-j\omega t}e^{j\omega t'}dtdt'$$

$$= \sum_{i=-I}^{I} A^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(t)p(t')e^{-j\omega(t+it_1)}e^{j\omega(t'+it_1)}dtdt' = \sum_{i=-I}^{I} A^2 |F_P(\omega)|^2 = 2IA^2|F_P(\omega)|^2$$

$$= \frac{2T}{t_1}A^2|F_P(\omega)|^2 \qquad \Rightarrow \quad S_X(\omega) = \lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T} = \frac{A^2|F_P(\omega)|^2}{t_1}$$

Spectral Density of Binary Processes

For $p(t) = p_r(t) = \text{rect}(t/t_1)$,

$$F_P(\omega) = t_1 \text{sinc}(t_1\omega/2\pi) \text{ and thus } |F_P(\omega)|^2 = t_1^2 \text{sinc}^2(t_1\omega/2\pi)$$

Therefore, $$S_X(\omega) = \frac{A^2 |F_P(\omega)|^2}{t_1} = A^2 t_1 \text{sinc}^2(t_1\omega/2\pi)$$

For $p(t) = p_c(t) = \frac{1}{2}\left(1 + \cos\frac{2\pi t}{t_1}\right), \quad |t| \le \frac{t_1}{2}; \ = 0, \quad |t| > \frac{t_1}{2},$

$$F_P(\omega) = \frac{1}{2}\int_{-t_1/2}^{t_1/2}\left(1 + \cos\frac{2\pi t}{t_1}\right)e^{-j\omega t}dt = \frac{t_1}{2}\left[\frac{\sin(\omega t_1/2)}{(\omega t_1/2)}\right]\left[\frac{\pi^2}{\pi^2 - (\omega t_1/2)^2}\right]$$

$$S_X(\omega) = \frac{A^2 t_1}{4}\left[\frac{\sin(\omega t_1/2)}{(\omega t_1/2)}\right]^2\left[\frac{\pi^2}{\pi^2 - (\omega t_1/2)^2}\right]^2$$

$$S_X(f) = \frac{A^2 t_1}{4}\text{sinc}^2(t_1 f)\left[\frac{1}{1-(t_1 f)^2}\right]^2$$

Note that in both cases, max $S_X(\omega)$ occurs at $\omega = 0$

Example

The spectral density of the pulses thus defines the bandwidth of the binary signal carried by these pulses. This bandwidth is a function of the pulse width t_1 .

We often need to "match" the bandwidth of the signal to the bandwidth of the transmission channel so as to reduce undesirable distortion of interference.

For example, we may request a bandwidth which would support transmission of the signal up to the frequency at which the spectral density is no more than 1% of its maximum. What would this bandwidth be?

$$\frac{S_X(f)}{S_X(0)} \le 0.01 \quad \text{for } |f| > f_1$$

For the rectangular pulse,

$$S_X(0) = A^2 t_1 \qquad S_X(f_1) = A^2 t_1 \text{sinc}^2(t_1 f_1) = 0.01 A^2 t_1 \Rightarrow \text{sinc}^2(t_1 f_1) = 0.01$$

$$t_1 f_1 \pi = 8.4226 \quad \Rightarrow \quad f_1 = 2.681 / t_1$$

For the raised cosine pulse,

$$S_X(0) = A^2 t_1 / 4$$

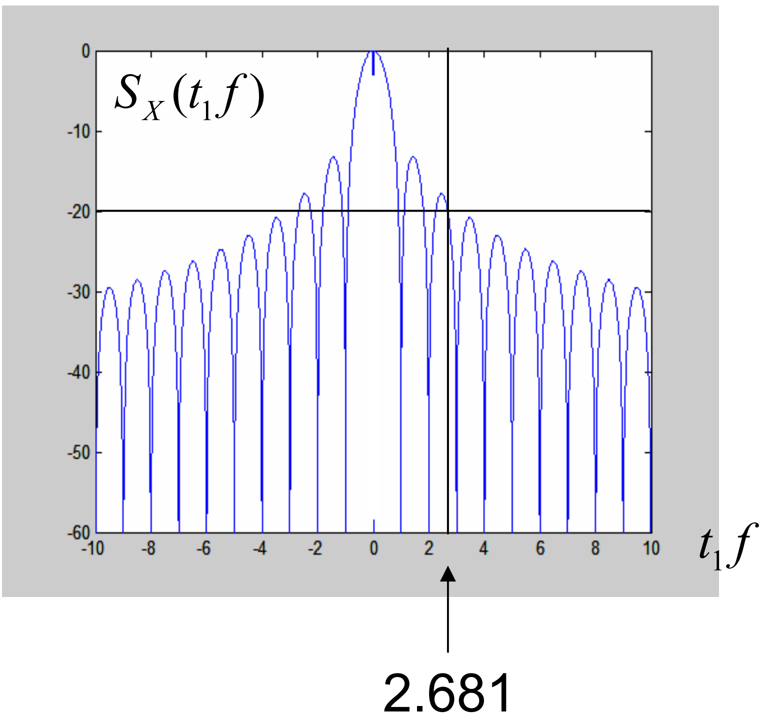
$$S_X(f_1) = \frac{A^2 t_1}{4} \text{sinc}^2(t_1 f_1) \left[\frac{1}{1-(t_1 f_1)^2} \right]^2 = 0.01 \frac{A^2 t_1}{4}$$

$$t_1 f_1 \pi = 5.1836 \quad \Rightarrow \quad f_1 = 1.65 / t_1$$

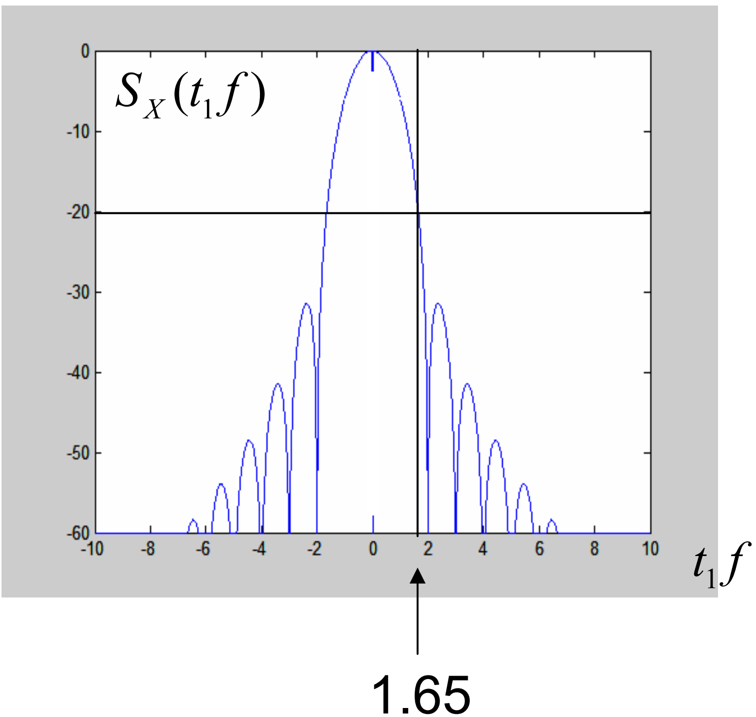
Bandwidth of Various Pulses

$$\frac{S_X(f)}{S_X(0)} \le 0.01 \quad \text{for } |f| > f_1 \;\Rightarrow\; 10\log_{10}(0.01) = -20\ (\text{dB}) \text{ at } f_1$$

Rectangular pulse



Raised-cosine pulse



Example

An nth- order Butterworth spectrum is one whose spectral density is given by

$$S_X(f) = \frac{1}{1+(f/W)^{2n}}$$

in which W is the so-called half-power bandwidth.

1. Find the bandwidth outside of which the spectral density is less than 1% of its maximum value.
2. For n=1, find the bandwidth (F) outside of which no more than 1% of the average power exists.

$S_X(f)$ is a monotonically decreasing function of f.

$$\max S_X = S_X(0) = \frac{1}{1+(0/W)^{2n}} = 1$$

$$S_X(f) = \frac{1}{1+(f/W)^{2n}} = 0.01 \quad \Rightarrow \quad 100 = 1+(f/W)^{2n} \quad \Rightarrow \quad f = W(99)^{1/(2n)}$$

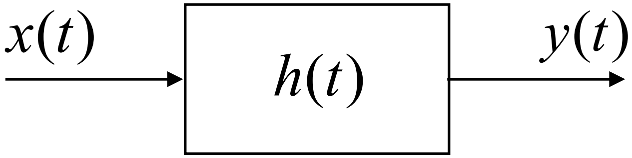
$$\int_{-\infty}^{\infty} S_X(f)df = \int_{-\infty}^{\infty} \frac{1}{1+(f/W)^2} df = W \tan^{-1}\left(\frac{f}{W}\right)\Bigg|_{-\infty}^{\infty} = W\pi$$

$$\int_p^q \frac{2a}{(2\pi f)^2 + a^2} df = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{2\pi f}{a}\right)\right]_p^q$$

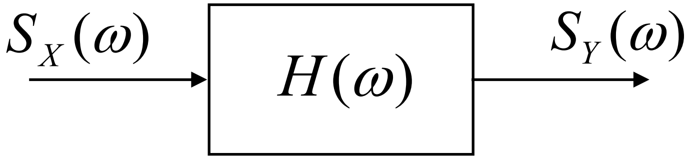
$$\int_{-F}^{F} \frac{1}{1+(f/W)^2} df = W \tan^{-1}\left(\frac{f}{W}\right)\Bigg|_{-F}^{F} = W2\tan^{-1}\left(\frac{F}{W}\right) = 0.99W\pi$$

$$F = 63.657W$$

System Analysis



Time-domain analysis



Frequency-domain analysis

- Only interested in bounded-input/bounded-output systems.
- The output of the system, $y(t)$, is the result of convolution between the input (excitation to the system), $x(t)$, and the **impulse response**, $h(t)$, of the system.

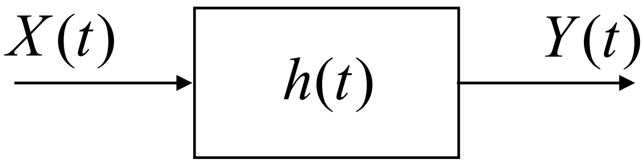
$$y(t) = \int_{-\infty}^{\infty} x(t-\lambda) h(\lambda) d\lambda$$

- For the system to be realizable and stable,

$$h(t) = 0 \text{ for } t < 0 \quad \text{(causality) and} \quad \int_{-\infty}^{\infty} |h(t)| dt < \infty \quad \text{(stability)}$$

Thus, $$y(t) = \int_{0}^{\infty} x(t-\lambda) h(\lambda) d\lambda = \int_{-\infty}^{0} x(\lambda) h(t-\lambda) d\lambda$$

Random Input to A System



$x(t) \Rightarrow X(t) \qquad y(t) \Rightarrow Y(t)$

The input is no longer a fixed function.

The same linear system concept applies.

Example:
$$h(t) = \begin{cases} 5e^{-3t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$X(t) = M + 4\cos(2t + \Theta)$ where M is a random variable and Θ is an independent random variable, uniformly distributed in $(0,2\pi)$.

$$Y(t) = \int_{-\infty}^{t} [M + 4\cos(2\lambda + \Theta)] 5e^{-3(t-\lambda)} d\lambda$$
$$= \frac{5}{3}M + \frac{20}{13}[3\cos(2t + \Theta) + 2\sin(2t + \Theta)]$$

$Y(t)$ is also a random process whose statistical properties can be derived from the distributions of the random variables, M and Θ .

Example

A linear system has an impulse response of the form

$$h(t) = \begin{cases} te^{-2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and an input that is a random process of the form

$$X(t) = M, \quad -\infty < t < \infty$$

where M is a random variable uniformly distributed in (0,12).

1. Write an expression for the output;
2. Find the mean value of the output;
3. Find the variance of the output.

$$Y(t) = \int_0^\infty M\lambda e^{-2\lambda} d\lambda = M \frac{e^{-2\lambda}}{4}(-2\lambda - 1)\Big|_0^\infty = \frac{M}{4}$$

$$E[Y(t)] = E[M / 4] = 6/4 = 3/2$$

$$E[Y^2(t)] = E[M^2 / 16] = (144/3)/16 = 3$$

$$\sigma_Y^2 = E[Y^2(t)] - \{E[Y(t)]\}^2 = 3 - (9/4) = 3/4$$

Example

A linear system has an impulse response of the form

$$h(t) = \begin{cases} 5\delta(t) + 3, & 0 \le t < 1 \\ 0, & \text{elsewhere} \end{cases}$$

The input process is of the form $X(t) = 2\cos(2\pi t + \Theta), \quad -\infty < t < \infty$ where Θ is a random variable uniformly distributed in $(0,2\pi)$.

1. Write an expression for the output process;
2. Find the mean value of the output;
3. Find the variance of the output.

$$Y(t) = \int_0^1 2\cos[2\pi(t-\lambda) + \Theta][5\delta(\lambda) + 3]d\lambda$$

$$= 10\cos(2\pi t + \Theta) - \frac{6}{2\pi}\sin[2\pi(t-\lambda) + \Theta]\Bigg|_{\lambda=0}^{1} = 10\cos(2\pi t + \Theta)$$

$$E[Y(t)] = E_\Theta[10\cos(2\pi t + \Theta)] = 0$$

$$E[Y^2(t)] = E_\Theta[100\cos^2(2\pi t + \Theta)] = E_\Theta[50 + 50\cos(4\pi t + 2\Theta)] = 50$$

$$\sigma_Y^2 = E[Y^2(t)] - \{E[Y(t)]\}^2 = 50 - 0 = 50$$

Mean of System Output

As demonstrated in the previous examples,

$$\overline{Y} = E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(t-\lambda)h(\lambda)d\lambda\right]$$

In general, for $E\left[\int_{t_1}^{t_2} Z(t)f(t)dt\right] = \int_{t_1}^{t_2} E[Z(t)]f(t)dt$, it requires that

1. $\int_{t_1}^{t_2} E[|Z(t)|]\,|f(t)|\,dt$
2. $Z(t)$ is bounded on the interval (t_1, t_2) .

In most cases, we assume these conditions are satisfied.

And if $X(t)$ is wide sense stationary with $E[X(t)] = \overline{X}$, then

$$\overline{Y} = \int_{-\infty}^{\infty} E[X(t-\lambda)]h(\lambda)d\lambda = \overline{X}\int_{-\infty}^{\infty} h(\lambda)d\lambda$$

Note that $\int_{-\infty}^{\infty} h(\lambda)d\lambda$ is the dc gain of the system; the dc component of the output is thus equal to the dc component of the input times the dc gain of the system.

Mean-Square Value of System Output

$$\overline{Y^2} = E[Y^2(t)] = E\left[\int_0^\infty X(t-\lambda_1)h(\lambda_1)d\lambda_1 \bullet \int_0^\infty X(t-\lambda_2)h(\lambda_2)d\lambda_2\right]$$

$$= E\left[\int_0^\infty d\lambda_1 \int_0^\infty X(t-\lambda_1)X(t-\lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_2\right]$$

$$= \int_0^\infty d\lambda_1 \int_0^\infty E[X(t-\lambda_1)X(t-\lambda_2)]h(\lambda_1)h(\lambda_2)d\lambda_2$$

But, $E[X(t-\lambda_1)X(t-\lambda_2)] = R_X(t-\lambda_1-t+\lambda_2) = R_X(\lambda_2-\lambda_1)$

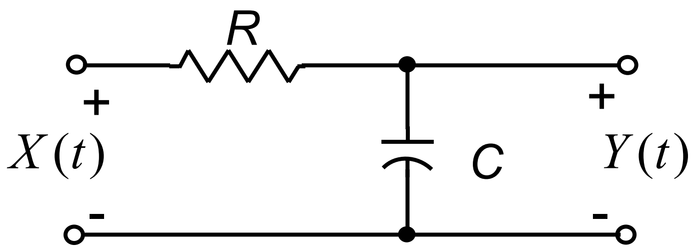
Therefore, $\overline{Y^2} = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2$

Example: For white noise input with $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$, the output "noise" power is

$$\overline{Y^2} = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2$$

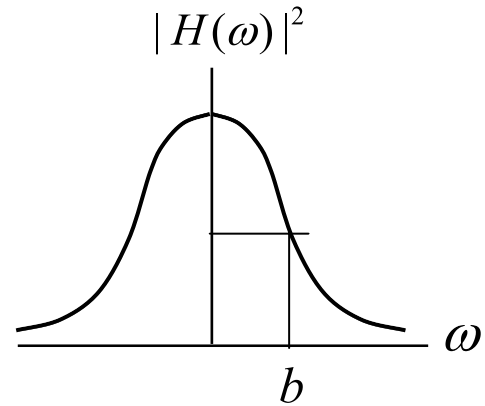
$$= \int_0^\infty d\lambda_1 \int_0^\infty \frac{N_0}{2}\delta(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2 = \frac{N_0}{2}\int_0^\infty h^2(\lambda)d\lambda$$

Analysis of A Simple System



$$h(t)=\begin{cases} be^{-bt}, & t\ge 0 \\ 0, & t<0 \end{cases}$$

where $b=\dfrac{1}{RC}$



$$\overline{Y}=\overline{X}\int_0^{\infty} be^{-b\lambda}d\lambda=\overline{X}\,\frac{be^{-b\lambda}}{-b}\Bigg|_0^{\infty}=\overline{X}$$

When the input is white noise with $R_X(\tau)=\dfrac{N_0}{2}\delta(\tau)$

$$\overline{Y^2}=\frac{N_0}{2}\int_0^{\infty} b^2e^{-2b\lambda}d\lambda=b^2\,\frac{N_0}{2}\,\frac{e^{-2b\lambda}}{-2b}\Bigg|_0^{\infty}=\frac{bN_0}{4}$$

$$\mathbf{F}\{h(t)\}=H(\omega)=\frac{b}{b+j\omega}\quad\text{(see Table A5);}\quad |H(\omega)|^2=\frac{b}{b+j\omega}\,\frac{b}{b-j\omega}=\frac{b^2}{b^2+\omega^2}$$

$|H(\omega=b)|^2=\dfrac{b^2}{2b^2}=\dfrac{1}{2}=\dfrac{1}{2}|H(\omega=0)|^2$ ⇨ b is the half-power bandwidth of the system

Let $W_{1/2}=\dfrac{b}{2\pi}\text{ Hz},\quad$ then $\quad\overline{Y^2}=\pi W_{1/2}\dfrac{N_0}{2}$

Example

A linear system has an impulse response $h(t) = te^{-2t}u(t)$ where u(t) is the unit step function. The input to this system is a white noise process having a 2-sided spectral density of $2 \ \mathrm{V}^2/\mathrm{Hz}$ plus a dc component of 2 V.

1. Find the mean value of the output of the system;
2. Find the variance of the output;
3. Find the mean-square value of the output.

$$X(t) = 2 + V(t) \text{ with } R_V(\tau) = 2\delta(\tau), \overline{X} = 2 + E[V(t)] = 2 \text{ and } R_X(\tau) = 4 + 2\delta(\tau)$$

$$\overline{Y} = \overline{X}\int_0^\infty te^{-2t}dt = 2\frac{e^{-2t}}{4}(-2t-1)\Bigg|_0^\infty = \frac{1}{2}$$

$$\overline{Y^2} = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2 = \int_0^\infty d\lambda_1 \int_0^\infty [4 + 2\delta(\lambda_2 - \lambda_1)]h(\lambda_1)h(\lambda_2)d\lambda_2$$

$$= 4\left[\int_0^\infty te^{-2t}dt\right]^2 + 2\int_0^\infty t^2 e^{-4t}dt = 4\left[\frac{e^{-2t}}{4}(-2t-1)\Bigg|_0^\infty\right]^2 + 2\frac{e^{-4t}}{-64}(16t^2 + 8t + 2)\Bigg|_0^\infty$$

$$= \frac{1}{4} + \frac{1}{16} = \frac{5}{16}$$

$$\sigma_Y^2 = \overline{Y^2} - \overline{Y}^2 = \frac{5}{16} - \frac{1}{4} = \frac{1}{16}$$