ECE 3075A

Random Signals

Lecture 22

Auto- and Cross-Correlation in System Analysis

School of Electrical and Computer Engineering
Georgia Institute of Technology
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Autocorrelation Function of System Output

$$R_Y(\tau) = E[Y(t)Y(t+\tau)]$$

$$R_Y(\tau) = E\left[\int_0^\infty d\lambda_1 \int_0^\infty X(t-\lambda_1)X(t+\tau-\lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_2\right]$$

$$= \int_0^\infty d\lambda_1 \int_0^\infty E[X(t-\lambda_1)X(t+\tau-\lambda_2)]h(\lambda_1)h(\lambda_2)d\lambda_2$$

$$= \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2-\lambda_1-\tau)h(\lambda_1)h(\lambda_2)d\lambda_2$$

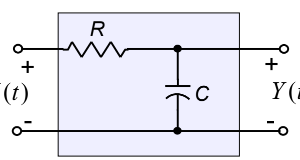
Note again: $R_Y(0) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2 = \overline{Y^2}$

For white noise input with $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$, the output "noise" autocorrelation function is

$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty \frac{N_0}{2}\delta(\lambda_2-\lambda_1-\tau)h(\lambda_1)h(\lambda_2)d\lambda_2 = \frac{N_0}{2}\int_0^\infty h(\lambda_1)h(\lambda_1+\tau)d\lambda_1$$

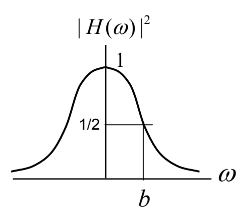
which is proportional to the **time correlation function** of the impulse response.

Analysis of A Simple System



$$h(t) = \begin{cases} be^{-bt}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

where $b = \frac{1}{RC}$



When the input is white noise with $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$

$$R_Y(\tau) = \frac{N_0}{2}\int_0^\infty be^{-b\lambda}be^{-b(\tau+\lambda)}d\lambda = b^2\frac{N_0}{2}e^{-b\tau}\left.\frac{e^{-2b\lambda}}{-2b}\right|_0^\infty = \frac{bN_0}{4}e^{-b\tau}, \quad \tau \ge 0$$

$$R_Y(\tau) = \frac{N_0}{2}\int_{-\tau}^\infty be^{-b\lambda}be^{-b(\tau+\lambda)}d\lambda = b^2\frac{N_0}{2}e^{-b\tau}\left.\frac{e^{-2b\lambda}}{-2b}\right|_{-\tau}^\infty = \frac{bN_0}{4}e^{+b\tau}, \quad \tau < 0$$

$R_Y(\tau) = \frac{bN_0}{4}e^{-b|\tau|}$ Recall b is the half-power bandwidth of the **system**.

$\mathbf{F}\{h(t)\} = H(\omega) = \frac{b}{b+j\omega}$ (see Table A5); $|H(\omega)|^2 = \frac{b}{b+j\omega}\frac{b}{b-j\omega} = \frac{b^2}{b^2+\omega^2}$

Revisit to Exponential Autocorrelation

$R_X(\tau) = Ae^{-\beta|\tau|} \quad A > 0, \beta > 0$ Signal!!

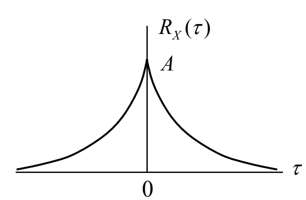
$$S_X(\omega) = \int_{-\infty}^0 Ae^{\beta\tau}e^{-j\omega\tau}d\tau + \int_0^\infty Ae^{-\beta\tau}e^{-j\omega\tau}d\tau = A\left.\frac{e^{(\beta-j\omega)\tau}}{\beta-j\omega}\right|_{-\infty}^0 + A\left.\frac{e^{-(\beta+j\omega)\tau}}{-(\beta+j\omega)}\right|_0^\infty$$

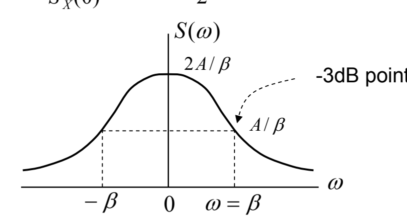
$$= A\left[\frac{1}{\beta-j\omega} + \frac{1}{\beta+j\omega}\right] = \frac{2\beta A}{\beta^2+\omega^2}$$

$S_X(\omega) = \frac{2\beta^2}{\beta^2+\omega^2}$ if $A = \beta$

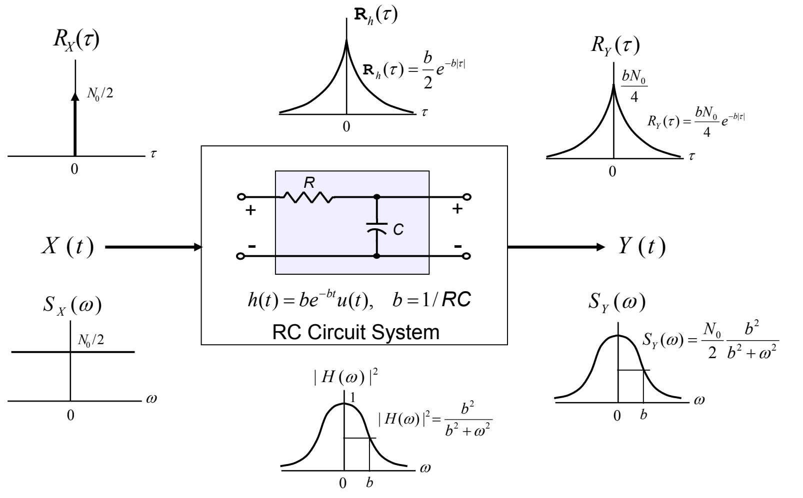
$$S_X(\omega=0) = \frac{2\beta A}{\beta^2+0^2} = \frac{2A}{\beta} \qquad S_X(\omega=\beta) = \frac{2\beta A}{\beta^2+\beta^2} = \frac{A}{\beta} = \frac{1}{2}S_X(0)$$

$\frac{S_X(\beta)}{S_X(0)} = \frac{1}{2}$ Or in dB: $10\log_{10}\frac{S_X(\beta)}{S_X(0)} = 10\log_{10}\frac{1}{2} = -3.0103 \text{ dB}$





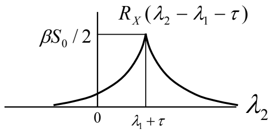
RC Circuit – Input Output Relationship



Non-White Input

Consider an input process to the previous RC circuit with autocorrelation function: $R_X(\tau)=\frac{\beta S_0}{2}e^{-\beta|\tau|}$

Recall the filtered noise!



When $\tau>0$

$$R_Y(\tau)=\int_0^\infty d\lambda_1\int_0^{\lambda_1+\tau}R_X(\lambda_2-\lambda_1-\tau)h(\lambda_1)h(\lambda_2)d\lambda_2$$

$$+\int_0^\infty d\lambda_1\int_{\lambda_1+\tau}^\infty R_X(\lambda_2-\lambda_1-\tau)h(\lambda_1)h(\lambda_2)d\lambda_2$$

$$=\frac{b^2\beta S_0}{2}\left[\int_0^\infty e^{-(b+\beta)\lambda_1}d\lambda_1\int_0^{\lambda_1+\tau}e^{-\beta\tau}e^{-(b-\beta)\lambda_2}d\lambda_2+\int_0^\infty e^{-(b-\beta)\lambda_1}d\lambda_1\int_{\lambda_1+\tau}^\infty e^{\beta\tau}e^{-(b+\beta)\lambda_2}d\lambda_2\right]$$

$$=\frac{b^2\beta S_0}{2(b-\beta)}\left(-\frac{e^{-b\tau}}{2b}+\frac{e^{-\beta\tau}}{b+\beta}\right)+\frac{b^2\beta S_0}{2(b+\beta)}\left(\frac{e^{-b\tau}}{2b}\right)=\frac{b^2\beta S_0}{2(b^2-\beta^2)}\left(e^{-\beta\tau}-\frac{\beta e^{-b\tau}}{b}\right)$$

Since $R_Y(\tau)$ is symmetric w.r.t. τ

$$R_Y(\tau)=\frac{b^2\beta S_0}{2(b^2-\beta^2)}\left(e^{-\beta|\tau|}-\frac{\beta e^{-b|\tau|}}{b}\right)$$

The Issue of Bandwidth

$$R_Y(\tau)=\frac{b^2\beta S_0}{2(b^2-\beta^2)}\left(e^{-\beta|\tau|}-\frac{\beta e^{-b|\tau|}}{b}\right)$$

$$\lim_{\beta\to\infty}R_Y(\tau)=\frac{b^2\beta^2 S_0}{2(b^2-\beta^2)}\left(-\frac{e^{-b|\tau|}}{b}\right)=\frac{bS_0}{2}e^{-b|\tau|}$$

When $\beta\to\infty$, $R_X(\tau)\to\delta(\tau)$ and $X(t)\to$ white noise.

Rearranging the terms in $R_Y(\tau)$, we have

$$R_Y(\tau)=\frac{bS_0}{2}e^{-b|\tau|}\left[\frac{1}{1-(b/\beta)^2}\right]\left[1-\frac{b}{\beta}e^{-(\beta-b)|\tau|}\right]$$

If $b<<\beta$, i.e. when the input bandwidth far exceeds the system bandwidth, the bracketed terms approach unity and $R_Y(\tau)$ approaches that of the output when the input is white noise.

Example

A linear system has an impulse response of $h(t)=4e^{-4t}u(t)$

The input to this system is a random process with autocorrelation function given by $R_X(\tau)=e^{-2|\tau|}$

Find the value of the autocorrelation function of the output of the system for: $a)\ \tau=0;\quad b)\ \tau=0.5;\quad c)\ \tau=1.$

$h(t)=be^{-bt}u(t)$ where $b=4$ $\qquad R_X(\tau)=\frac{\beta S_0}{2}e^{-\beta|\tau|}=e^{-2|\tau|}\ \Rightarrow\ \beta=2, S_0=1$

$$R_Y(\tau)=\frac{16\times 2}{2(16-4)}\left(e^{-2|\tau|}-\frac{e^{-4|\tau|}}{2}\right)=\frac{4}{3}\left(e^{-2|\tau|}-\frac{e^{-4|\tau|}}{2}\right)$$

$$R_Y(\tau=0)=\frac{4}{3}\left(e^{-2|0|}-\frac{e^{-4|0|}}{2}\right)=\frac{4}{3}\left(1-\frac{1}{2}\right)=\frac{2}{3}$$

$$R_Y(\tau=0.5)=\frac{4}{3}\left(e^{-1}-\frac{e^{-2}}{2}\right)=\frac{4}{3}(0.3679-0.0676)\approx 0.4$$

$$R_Y(\tau=1)=\frac{4}{3}\left(e^{-2}-\frac{e^{-4}}{2}\right)=\frac{4}{3}(0.1353-0.0092)=0.1682$$

Input-Output Crosscorrelation

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E\left[X(t)\int_0^\infty X(t+\tau-\lambda)h(\lambda)d\lambda\right]$$

$$= \int_0^\infty E[X(t)X(t+\tau-\lambda)]h(\lambda)d\lambda$$

$$= \int_0^\infty R_X(\tau-\lambda)h(\lambda)d\lambda$$

The crosscorrelation function is the convolution of the input auto-correlation and the impulse response of the system!!

$$R_{YX}(\tau) = E[X(t+\tau)Y(t)] = \int_0^\infty E[X(t+\tau)X(t-\lambda)]h(\lambda)d\lambda$$

$$= \int_0^\infty R_X(\tau+\lambda)h(\lambda)d\lambda$$

Note: $R_{XY}(-\tau) = \int_0^\infty R_X(-\tau-\lambda)h(\lambda)d\lambda = \int_0^\infty R_X(\tau+\lambda)h(\lambda)d\lambda = R_{YX}(\tau)$

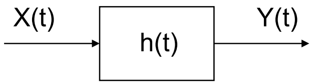
Crosscorrelation is useful in finding the impulse response of a linear system.

Crosscorrelation with White Noise Input

Consider a white noise input signal whose autocorrelation is

$$R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

$$Y(t) = \int_0^\infty X(t-\lambda)h(\lambda)d\lambda$$



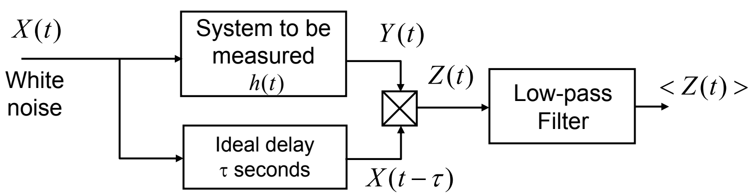
$$R_{XY}(\tau) = \int_0^\infty R_X(\tau-\lambda)h(\lambda)d\lambda$$

$$= \int_0^\infty \frac{N_0}{2}\delta(\tau-\lambda)h(\lambda)d\lambda = \begin{cases} \frac{N_0}{2}h(\tau), & \tau \ge 0 \\ 0, & \tau < 0 \end{cases}$$

Similarly, $R_{YX}(\tau) = \int_0^\infty \frac{N_0}{2}\delta(\tau+\lambda)h(\lambda)d\lambda = \begin{cases} 0, & \tau > 0 \\ \frac{N_0}{2}h(-\tau), & \tau < 0 \end{cases}$

The crosscorrelation function is proportional to the impulse response!

Method for Measuring Impulse Response



$$Y(t) = \int_0^\infty X(t-\lambda)h(\lambda)d\lambda$$

$Z(t) = X(t-\tau)Y(t)$ and $E[Z(t)] = E[X(t-\tau)Y(t)] = R_{XY}(\tau)$

A properly designed low-pass filter will be able to produce a time average of $Z(t)$, i.e., $< Z(t) >$ which is equal to $E[Z(t)]$ for an ergodic process.

$$< Z(t) > \cong R_{XY}(\tau) = \frac{N_0}{2}h(\tau) \quad \text{for } \tau \ge 0$$

An Alternative Expression of Output Autocorrelation

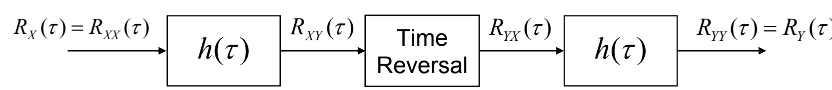
$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2-\lambda_1-\tau)h(\lambda_1)h(\lambda_2)d\lambda_2$$

$$= \int_0^\infty \left\{\int_0^\infty R_X(\lambda_2-\tau-\lambda_1)h(\lambda_1)d\lambda_1\right\}h(\lambda_2)d\lambda_2 = \int_0^\infty R_{XY}(\lambda_2-\tau)h(\lambda_2)d\lambda_2$$

In the above, $R_{XY}(\lambda_2-\tau) = \int_0^\infty R_X(\lambda_2-\tau-\lambda_1)h(\lambda_1)d\lambda_1$

Or in general form, $R_{XY}(\tau) = \int_0^\infty R_X(\tau-\lambda)h(\lambda)d\lambda$

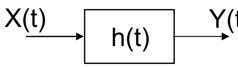
And $R_Y(\tau) = R_{YY}(\tau) = \int_0^\infty R_{XY}(\lambda-\tau)h(\lambda)d\lambda = \int_0^\infty R_{YX}(\tau-\lambda)h(\lambda)d\lambda$

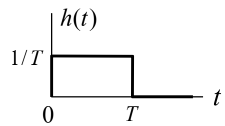


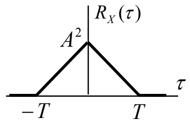
Calculation of output autocorrelation via convolution.

Time Domain Analysis - Example

Consider a finite-time integrator which has an impulse response







Input autocorrelation

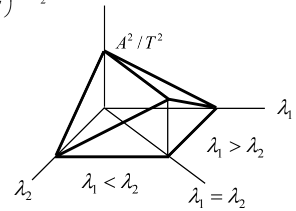
$$h(t) = \frac{1}{T}[u(t) - u(t-T)]$$

$$\overline{Y} = \overline{X}\int_0^T \frac{1}{T}dt = \overline{X}$$

$$\overline{Y^2} = \int_0^T d\lambda_1 \int_0^T R_X(\lambda_2 - \lambda_1)\left(\frac{1}{T}\right)^2 d\lambda_2 = \int_0^T d\lambda_1 \int_0^T A^2\left[1 - \frac{|\lambda_2 - \lambda_1|}{T}\right]\left(\frac{1}{T}\right)^2 d\lambda_2$$

$$= \frac{A^2}{T^2}\left\{\int_0^T\int_0^{\lambda_2}\left[1 - \frac{\lambda_2 - \lambda_1}{T}\right]d\lambda_1 d\lambda_2 + \int_0^T\int_0^{\lambda_1}\left[1 - \frac{\lambda_1 - \lambda_2}{T}\right]d\lambda_2 d\lambda_1\right\}$$

$$= \frac{A^2}{T^2}\left(\frac{T^2}{3} + \frac{T^2}{3}\right) = \frac{2A^2}{3}$$



In a similar manner, one can find the autocorrelation function of the output.

Finite Time Average of White Noise

Now, if the input is zero mean white noise, $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$

$$\overline{Y} = \overline{X}\int_0^T \frac{1}{T}dt = \overline{X} = 0$$

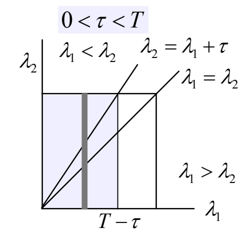
$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau)h(\lambda_1)h(\lambda_2)d\lambda_2$$

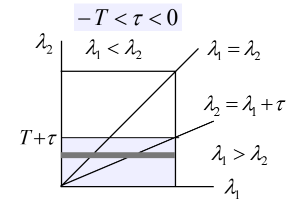
$$= \frac{N_0}{2}\int_0^T d\lambda_1 \int_0^T \delta(\lambda_2 - \lambda_1 - \tau)\frac{1}{T}\frac{1}{T}d\lambda_2$$

$$\int_0^T\int_0^T \delta(\lambda_2 - \lambda_1 - \tau)d\lambda_1 d\lambda_2$$

$$= \begin{cases} \int_0^{T-\tau}\int_0^T \delta(\lambda_2 - \lambda_1 - \tau)d\lambda_2 d\lambda_1, & 0 < \tau < T \\ \int_0^{T+\tau}\int_0^T \delta(\lambda_2 - \lambda_1 - \tau)d\lambda_1 d\lambda_2, & -T < \tau < 0 \end{cases}$$

$$= T - |\tau|$$





Therefore, $R_Y(\tau) = \frac{N_0}{2T}\left(1 - \frac{|\tau|}{T}\right)$

And, $\overline{Y^2} = R_Y(\tau = 0) = \frac{N_0}{2T}$

Analysis of "Signal Plus Noise"

Consider a signal/process $X(t) = A + V(t)$ where the noise $V(t)$ has an autocorrelation function of $R_V(\tau) = 10e^{-1000|\tau|}$.
An RC circuit is being used to filter out the noise and the requirement is to measure A with an error of 1% when A is on the order of 1. Determine the RC time constant.

For the noise: $R_V(\tau) = \frac{\beta S_0}{2}e^{-\beta|\tau|} = 10e^{-1000|\tau|}$ $\beta = 1000, S_0 = 0.02$
A wideband noise

$S_V(\omega) = \int_{-\infty}^{\infty} R_V(\tau)e^{-j\omega\tau}d\tau$ and $S_V(0) = \int_{-\infty}^{\infty} R_V(\tau)d\tau = 2\int_0^\infty 10e^{-1000\tau}d\tau = 0.02$

around $\omega = 0$, $S_V(\omega) \approx S_V(0) = 0.02$

$Y(t) = A + U(t)$ where $U(t)$ is the filtered version of $V(t)$

$$\overline{U^2} = R_U(0) \approx \frac{bN_0}{4} = \frac{bS_V(0)}{2} = 0.01b, \quad \text{or} \quad \sqrt{\overline{U^2}} \approx 0.1\sqrt{b}$$

The requirement dictates that $\sqrt{\overline{U^2}} \approx 0.1\sqrt{b} \le 1\% = 0.01 \Rightarrow b \le 0.01$

Since $b = 1/RC \Rightarrow RC \ge 100$