

ECE 3075A
Random Signals

Lecture 22

Auto- and Cross-Correlation in System Analysis

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Autocorrelation Function of System Output

$$R_Y(\tau) = E[Y(t)Y(t + \tau)]$$

$$\begin{aligned} R_Y(\tau) &= E\left[\int_0^\infty d\lambda_1 \int_0^\infty X(t - \lambda_1)X(t + \tau - \lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_2\right] \\ &= \int_0^\infty d\lambda_1 \int_0^\infty E[X(t - \lambda_1)X(t + \tau - \lambda_2)]h(\lambda_1)h(\lambda_2)d\lambda_2 \\ &= \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau)h(\lambda_1)h(\lambda_2)d\lambda_2 \end{aligned}$$

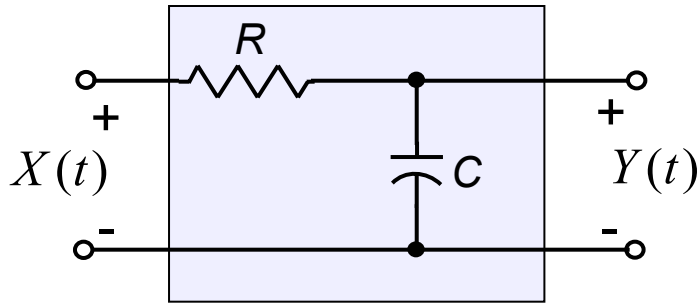
Note again: $R_Y(0) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2 = \overline{Y^2}$

For white noise input with $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$, the output “noise” autocorrelation function is

$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty \frac{N_0}{2}\delta(\lambda_2 - \lambda_1 - \tau)h(\lambda_1)h(\lambda_2)d\lambda_2 = \frac{N_0}{2} \int_0^\infty h(\lambda_1)h(\lambda_1 + \tau)d\lambda_1$$

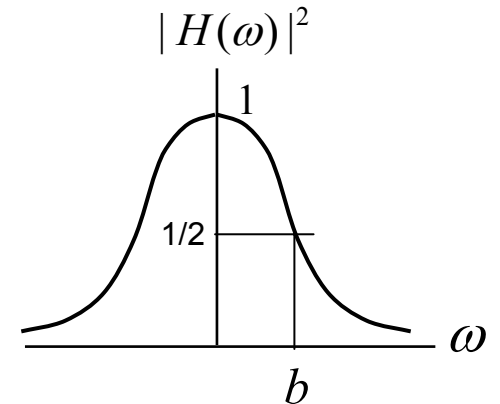
which is proportional to the **time correlation function** of the impulse response.

Analysis of A Simple System



$$h(t) = \begin{cases} be^{-bt}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

where $b = \frac{1}{RC}$



When the input is white noise with $R_X(\tau) = \frac{N_0}{2} \delta(\tau)$

$$R_Y(\tau) = \frac{N_0}{2} \int_0^{\infty} be^{-b\lambda} be^{-b(\tau+\lambda)} d\lambda = b^2 \frac{N_0}{2} e^{-b\tau} \frac{e^{-2b\lambda}}{-2b} \Big|_0^{\infty} = \frac{bN_0}{4} e^{-b\tau}, \quad \tau \geq 0$$

$$R_Y(\tau) = \frac{N_0}{2} \int_{-\tau}^{\infty} be^{-b\lambda} be^{-b(\tau+\lambda)} d\lambda = b^2 \frac{N_0}{2} e^{-b\tau} \frac{e^{-2b\lambda}}{-2b} \Big|_{-\tau}^{\infty} = \frac{bN_0}{4} e^{+b\tau}, \quad \tau < 0$$

$$R_Y(\tau) = \frac{bN_0}{4} e^{-b|\tau|}$$

Recall b is the half-power bandwidth of the **system**.

$$\mathbf{F}\{h(t)\} = H(\omega) = \frac{b}{b + j\omega} \quad (\text{see Table A5}); \quad |H(\omega)|^2 = \frac{b}{b + j\omega} \frac{b}{b - j\omega} = \frac{b^2}{b^2 + \omega^2}$$

Revisit to Exponential Autocorrelation

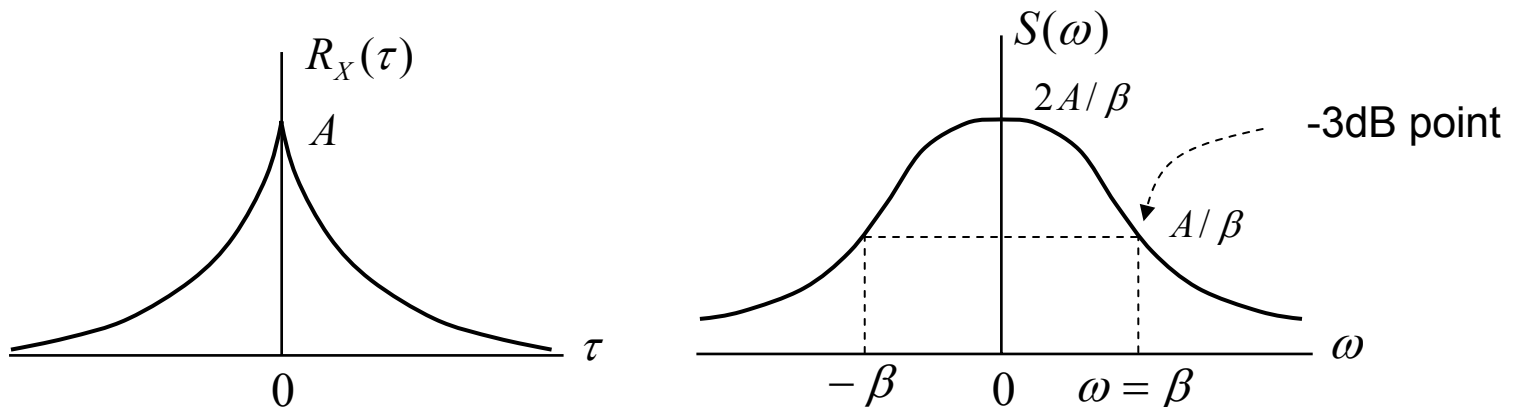
$$R_X(\tau) = Ae^{-\beta|\tau|} \quad A > 0, \beta > 0 \quad \text{Signal!!}$$

$$S_X(\omega) = \int_{-\infty}^0 Ae^{\beta\tau} e^{-j\omega\tau} d\tau + \int_0^{\infty} Ae^{-\beta\tau} e^{-j\omega\tau} d\tau = A \frac{e^{(\beta-j\omega)\tau}}{\beta-j\omega} \Big|_{-\infty}^0 + A \frac{e^{-(\beta+j\omega)\tau}}{-(\beta+j\omega)} \Big|_0^{\infty}$$

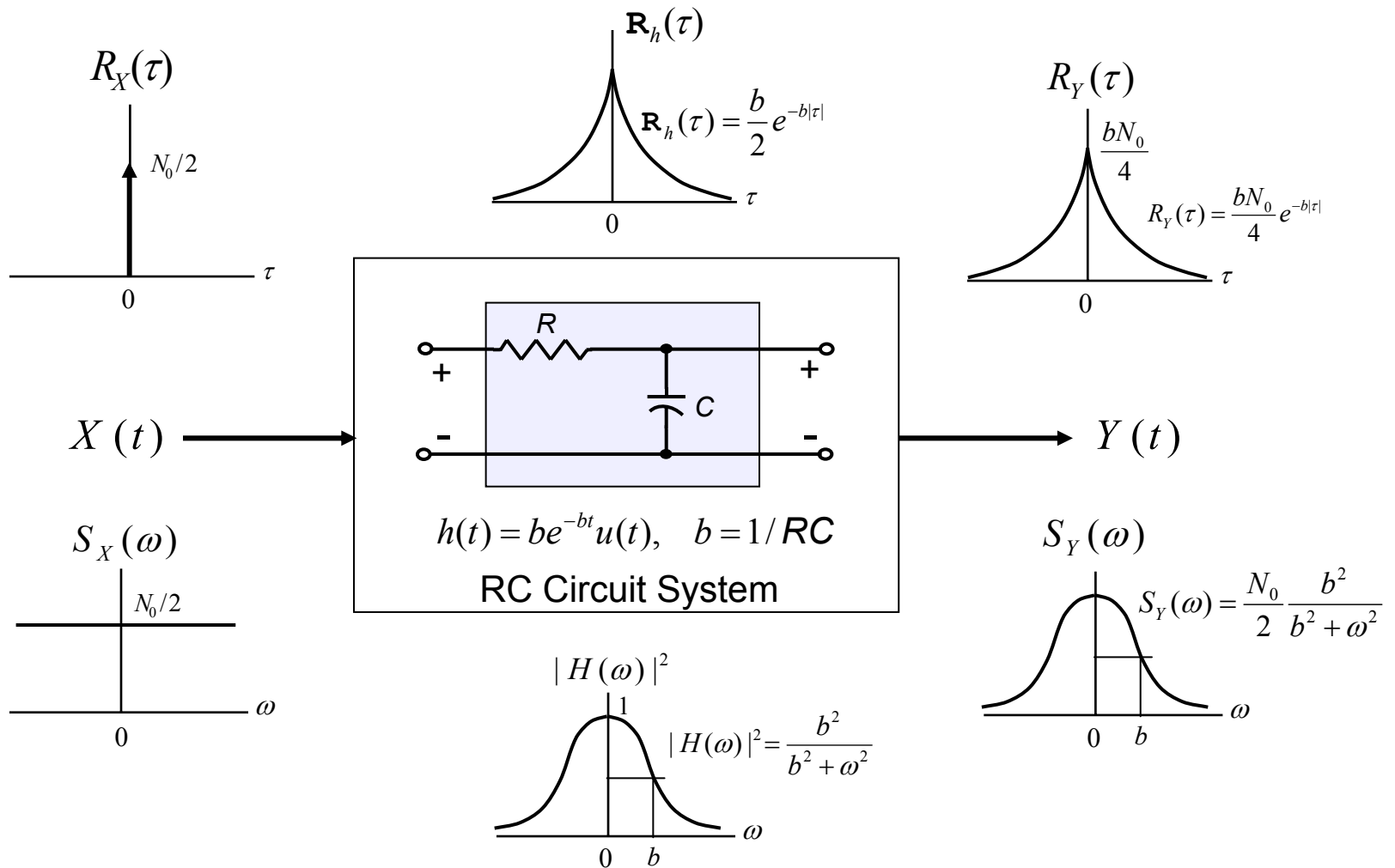
$$= A \left[\frac{1}{\beta-j\omega} + \frac{1}{\beta+j\omega} \right] = \frac{2\beta A}{\beta^2 + \omega^2} \quad S_X(\omega) = \frac{2\beta^2}{\beta^2 + \omega^2} \text{ if } A = \beta$$

$$S_X(\omega=0) = \frac{2\beta A}{\beta^2 + 0^2} = \frac{2A}{\beta} \quad S_X(\omega=\beta) = \frac{2\beta A}{\beta^2 + \beta^2} = \frac{A}{\beta} = \frac{1}{2} S_X(0)$$

$$\frac{S_X(\beta)}{S_X(0)} = \frac{1}{2} \quad \text{Or in dB: } 10 \log_{10} \frac{S_X(\beta)}{S_X(0)} = 10 \log_{10} \frac{1}{2} = -3.0103 \text{ dB}$$



RC Circuit – Input Output Relationship



Non-White Input

Consider an input process to the previous RC circuit with autocorrelation function:

$$R_X(\tau) = \frac{\beta S_0}{2} e^{-\beta|\tau|}$$

Recall the filtered noise!

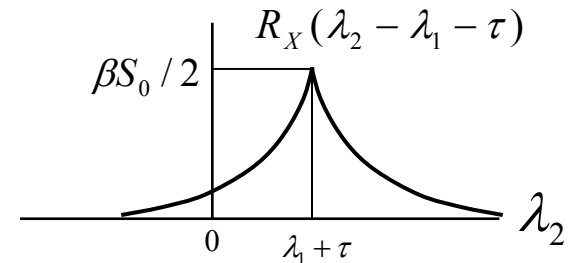
When $\tau > 0$

$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^{\lambda_1+\tau} R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2$$

$$+ \int_0^\infty d\lambda_1 \int_{\lambda_1+\tau}^\infty R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2$$

$$= \frac{b^2 \beta S_0}{2} \left[\int_0^\infty e^{-(b+\beta)\lambda_1} d\lambda_1 \int_0^{\lambda_1+\tau} e^{-\beta\tau} e^{-(b-\beta)\lambda_2} d\lambda_2 + \int_0^\infty e^{-(b-\beta)\lambda_1} d\lambda_1 \int_{\lambda_1+\tau}^\infty e^{\beta\tau} e^{-(b+\beta)\lambda_2} d\lambda_2 \right]$$

$$= \frac{b^2 \beta S_0}{2(b-\beta)} \left(-\frac{e^{-b\tau}}{2b} + \frac{e^{-\beta\tau}}{b+\beta} \right) + \frac{b^2 \beta S_0}{2(b+\beta)} \left(\frac{e^{-b\tau}}{2b} \right) = \frac{b^2 \beta S_0}{2(b^2 - \beta^2)} \left(e^{-\beta\tau} - \frac{\beta e^{-b\tau}}{b} \right)$$



Since $R_Y(\tau)$ is symmetric w.r.t. τ

$$R_Y(\tau) = \frac{b^2 \beta S_0}{2(b^2 - \beta^2)} \left(e^{-\beta|\tau|} - \frac{\beta e^{-b|\tau|}}{b} \right)$$

The Issue of Bandwidth

$$R_Y(\tau) = \frac{b^2 \beta S_0}{2(b^2 - \beta^2)} \left(e^{-\beta|\tau|} - \frac{\beta e^{-b|\tau|}}{b} \right)$$

$$\lim_{\beta \rightarrow \infty} R_Y(\tau) = \frac{b^2 \beta^2 S_0}{2(b^2 - \beta^2)} \left(-\frac{e^{-b|\tau|}}{b} \right) = \frac{b S_0}{2} e^{-b|\tau|}$$

When $\beta \rightarrow \infty$, $R_X(\tau) \rightarrow \delta(\tau)$ and $X(t) \rightarrow$ white noise.

Rearranging the terms in $R_Y(\tau)$, we have

$$R_Y(\tau) = \frac{b S_0}{2} e^{-b|\tau|} \left[\frac{1}{1 - (b/\beta)^2} \right] \left[1 - \frac{b}{\beta} e^{-(\beta-b)|\tau|} \right]$$

If $b \ll \beta$, i.e. when the input bandwidth far exceeds the system bandwidth, the bracketed terms approach unity and $R_Y(\tau)$ approaches that of the output when the input is white noise.

Example

A linear system has an impulse response of $h(t) = 4e^{-4t}u(t)$

The input to this system is a random process with autocorrelation function given by $R_X(\tau) = e^{-2|\tau|}$

Find the value of the autocorrelation function of the output of the system for: a) $\tau = 0$; b) $\tau = 0.5$; c) $\tau = 1$.

$$h(t) = be^{-bt}u(t) \text{ where } b = 4 \quad R_X(\tau) = \frac{\beta S_0}{2} e^{-\beta|\tau|} = e^{-2|\tau|} \Rightarrow \beta = 2, S_0 = 1$$

$$R_Y(\tau) = \frac{16 \times 2}{2(16 - 4)} \left(e^{-2|\tau|} - \frac{e^{-4|\tau|}}{2} \right) = \frac{4}{3} \left(e^{-2|\tau|} - \frac{e^{-4|\tau|}}{2} \right)$$

$$R_Y(\tau = 0) = \frac{4}{3} \left(e^{-2|0|} - \frac{e^{-4|0|}}{2} \right) = \frac{4}{3} \left(1 - \frac{1}{2} \right) = \frac{2}{3}$$

$$R_Y(\tau = 0.5) = \frac{4}{3} \left(e^{-1} - \frac{e^{-2}}{2} \right) = \frac{4}{3} (0.3679 - 0.0676) \approx 0.4$$

$$R_Y(\tau = 1) = \frac{4}{3} \left(e^{-2} - \frac{e^{-4}}{2} \right) = \frac{4}{3} (0.1353 - 0.0092) = 0.1682$$

Input-Output Crosscorrelation

$$\begin{aligned}R_{XY}(\tau) &= E[X(t)Y(t+\tau)] = E\left[X(t)\int_0^\infty X(t+\tau-\lambda)h(\lambda)d\lambda\right] \\ &= \int_0^\infty E[X(t)X(t+\tau-\lambda)]h(\lambda)d\lambda \\ &= \int_0^\infty R_X(\tau-\lambda)h(\lambda)d\lambda\end{aligned}$$

The crosscorrelation function is the convolution of the input autocorrelation and the impulse response of the system!!

$$\begin{aligned}R_{YX}(\tau) &= E[X(t+\tau)Y(t)] = \int_0^\infty E[X(t+\tau)X(t-\lambda)]h(\lambda)d\lambda \\ &= \int_0^\infty R_X(\tau+\lambda)h(\lambda)d\lambda\end{aligned}$$

Note: $R_{XY}(-\tau) = \int_0^\infty R_X(-\tau-\lambda)h(\lambda)d\lambda = \int_0^\infty R_X(\tau+\lambda)h(\lambda)d\lambda = R_{YX}(\tau)$

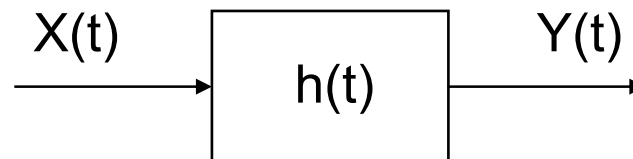
Crosscorrelation is useful in finding the impulse response of a linear system.

Crosscorrelation with White Noise Input

Consider a white noise input signal whose autocorrelation is

$$R_X(\tau) = \frac{N_0}{2} \delta(\tau)$$

$$Y(t) = \int_0^\infty X(t - \lambda)h(\lambda)d\lambda$$



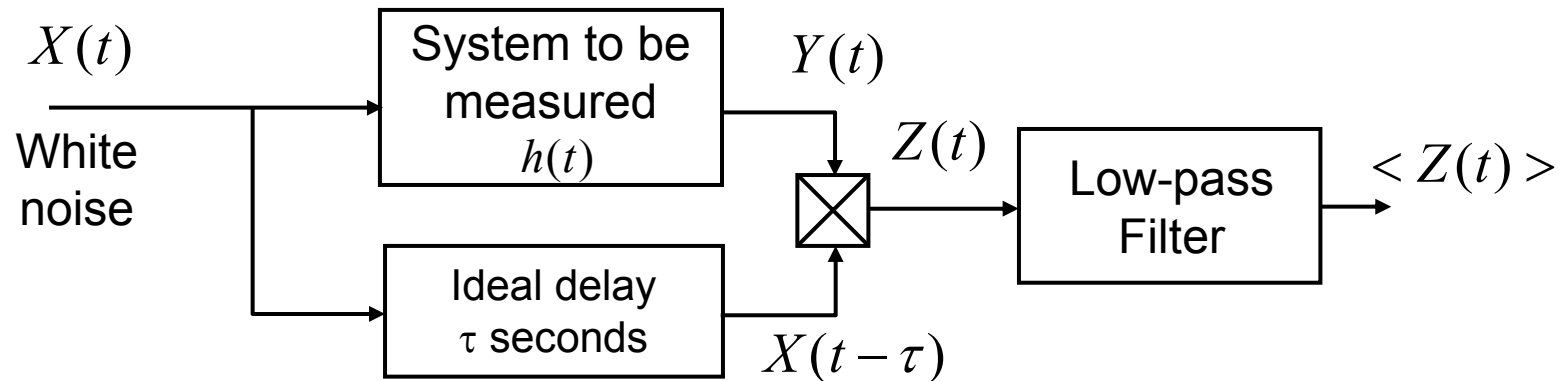
$$R_{XY}(\tau) = \int_0^\infty R_X(\tau - \lambda)h(\lambda)d\lambda$$

$$= \int_0^\infty \frac{N_0}{2} \delta(\tau - \lambda)h(\lambda)d\lambda = \begin{cases} \frac{N_0}{2} h(\tau), & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}$$

$$\text{Similarly, } R_{YX}(\tau) = \int_0^\infty \frac{N_0}{2} \delta(\tau + \lambda)h(\lambda)d\lambda = \begin{cases} 0, & \tau > 0 \\ \frac{N_0}{2} h(-\tau), & \tau < 0 \end{cases}$$

The crosscorrelation function is proportional to the impulse response!

Method for Measuring Impulse Response



$$Y(t) = \int_0^{\infty} X(t - \lambda)h(\lambda)d\lambda$$

$$Z(t) = X(t - \tau)Y(t) \text{ and } E[Z(t)] = E[X(t - \tau)Y(t)] = R_{XY}(\tau)$$

A properly designed low-pass filter will be able to produce a time average of $Z(t)$, i.e., $\langle Z(t) \rangle$ which is equal to $E[Z(t)]$ for an ergodic process.

$$\langle Z(t) \rangle \cong R_{XY}(\tau) = \frac{N_0}{2}h(\tau) \text{ for } \tau \geq 0$$

An Alternative Expression of Output Autocorrelation

$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2$$

$$= \int_0^\infty \left\{ \int_0^\infty R_X(\lambda_2 - \tau - \lambda_1) h(\lambda_1) d\lambda_1 \right\} h(\lambda_2) d\lambda_2 = \int_0^\infty R_{XY}(\lambda_2 - \tau) h(\lambda_2) d\lambda_2$$

In the above, $R_{XY}(\lambda_2 - \tau) = \int_0^\infty R_X(\lambda_2 - \tau - \lambda_1) h(\lambda_1) d\lambda_1$

Or in general form, $R_{XY}(\tau) = \int_0^\infty R_X(\tau - \lambda) h(\lambda) d\lambda$

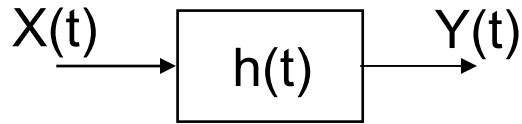
And $R_Y(\tau) = R_{YY}(\tau) = \int_0^\infty R_{XY}(\lambda - \tau) h(\lambda) d\lambda = \int_0^\infty R_{YX}(\tau - \lambda) h(\lambda) d\lambda$



Calculation of output autocorrelation via convolution.

Time Domain Analysis - Example

Consider a finite-time integrator which has an impulse response

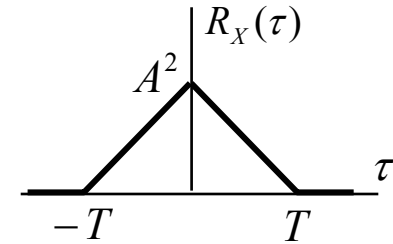
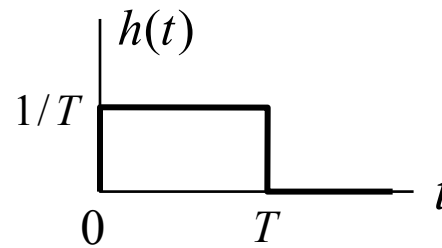


$$h(t) = \frac{1}{T} [u(t) - u(t - T)]$$

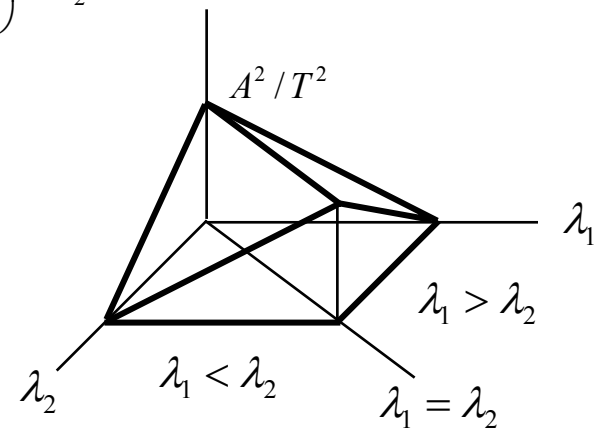
$$\bar{Y} = \bar{X} \int_0^T \frac{1}{T} dt = \bar{X}$$

$$\begin{aligned} \overline{Y^2} &= \int_0^T d\lambda_1 \int_0^T R_X(\lambda_2 - \lambda_1) \left(\frac{1}{T}\right)^2 d\lambda_2 = \int_0^T d\lambda_1 \int_0^T A^2 \left[1 - \frac{|\lambda_2 - \lambda_1|}{T}\right] \left(\frac{1}{T}\right)^2 d\lambda_2 \\ &= \frac{A^2}{T^2} \left\{ \int_0^T \int_0^{\lambda_2} \left[1 - \frac{\lambda_2 - \lambda_1}{T}\right] d\lambda_1 d\lambda_2 + \int_0^T \int_0^{\lambda_1} \left[1 - \frac{\lambda_1 - \lambda_2}{T}\right] d\lambda_2 d\lambda_1 \right\} \\ &= \frac{A^2}{T^2} \left(\frac{T^2}{3} + \frac{T^2}{3} \right) = \frac{2A^2}{3} \end{aligned}$$

In a similar manner, one can find the autocorrelation function of the output.



Input
autocorrelation



Finite Time Average of White Noise

Now, if the input is zero mean white noise, $R_X(\tau) = \frac{N_0}{2} \delta(\tau)$

$$\bar{Y} = \bar{X} \int_0^T \frac{1}{T} dt = \bar{X} = 0$$

$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2$$

$$= \frac{N_0}{2} \int_0^T d\lambda_1 \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) \frac{1}{T} \frac{1}{T} d\lambda_2$$

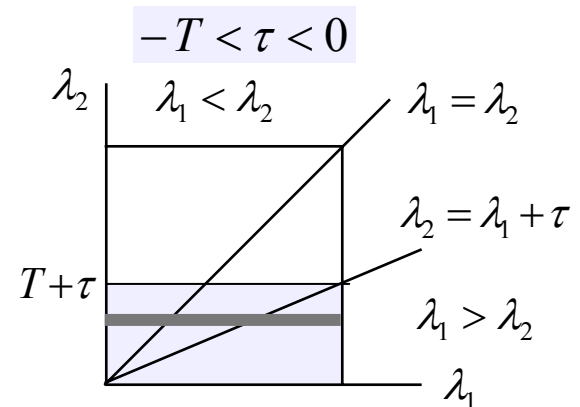
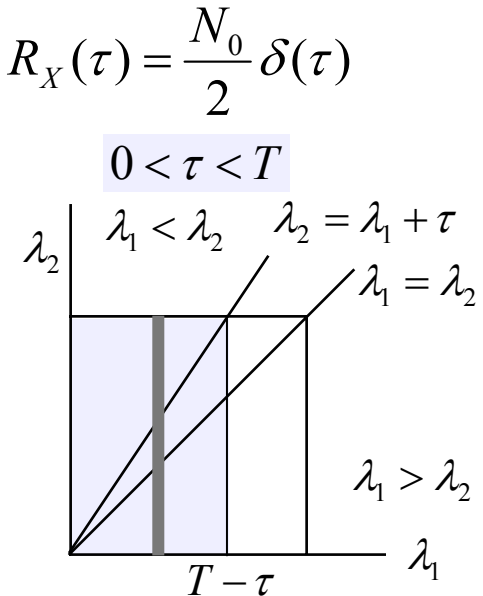
$$\int_0^T \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_1 d\lambda_2$$

$$= \begin{cases} \int_0^{T-\tau} \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_2 d\lambda_1, & 0 < \tau < T \\ \int_0^{T+\tau} \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_1 d\lambda_2, & -T < \tau < 0 \end{cases}$$

$$= T - |\tau|$$

Therefore, $R_Y(\tau) = \frac{N_0}{2T} \left(1 - \frac{|\tau|}{T} \right)$

And, $\overline{Y^2} = R_Y(\tau = 0) = \frac{N_0}{2T}$



Analysis of “Signal Plus Noise”

Consider a signal/process $X(t) = A + V(t)$ where the noise $V(t)$ has an autocorrelation function of $R_V(\tau) = 10e^{-1000|\tau|}$.

An RC circuit is being used to filter out the noise and the requirement is to measure A with an error of 1% when A is on the order of 1. Determine the RC time constant.

For the noise: $R_V(\tau) = \frac{\beta S_0}{2} e^{-\beta|\tau|} = 10e^{-1000|\tau|}$ $\beta = 1000, S_0 = 0.02$
A wideband noise

$$S_V(\omega) = \int_{-\infty}^{\infty} R_V(\tau) e^{-j\omega\tau} d\tau \quad \text{and} \quad S_V(0) = \int_{-\infty}^{\infty} R_V(\tau) d\tau = 2 \int_0^{\infty} 10e^{-1000\tau} d\tau = 0.02$$

around $\omega = 0$, $S_V(\omega) \approx S_V(0) = 0.02$

$Y(t) = A + U(t)$ where $U(t)$ is the filtered version of $V(t)$

$$\overline{U^2} = R_U(0) \approx \frac{bN_0}{4} = \frac{bS_V(0)}{2} = 0.01b, \quad \text{or} \quad \sqrt{\overline{U^2}} \approx 0.1\sqrt{b}$$

The requirement dictates that $\sqrt{\overline{U^2}} \approx 0.1\sqrt{b} \leq 1\% = 0.01 \Rightarrow b \leq 0.01$

Since $b = 1/RC \Rightarrow RC \geq 100$