

ECE 3075A  
Random Signals

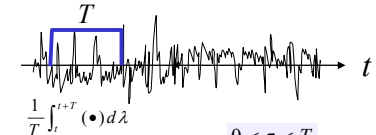
Lecture 23

System Analysis in Time & Frequency Domain

School of Electrical and Computer Engineering  
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Finite Time Average of White Noise

The input is zero mean white noise,  $R_X(\tau) = \frac{N_0}{2} \delta(\tau)$



$$\bar{Y} = \bar{X} \int_0^T \frac{1}{T} dt = \bar{X} = 0$$

$$R_Y(\tau) = \int_0^{\infty} d\lambda_1 \int_0^{\infty} R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2$$

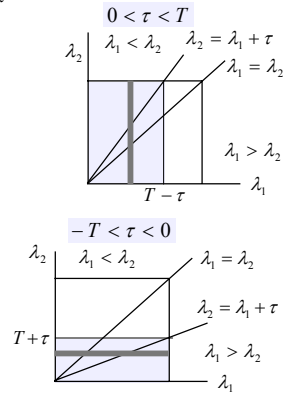
$$= \frac{N_0}{2} \int_0^T d\lambda_1 \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) \frac{1}{T} \frac{1}{T} d\lambda_2$$

$$\int_0^T \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_1 d\lambda_2$$

$$= \begin{cases} \int_0^{T-\tau} \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_2 d\lambda_1, & 0 < \tau < T \\ \int_0^{T+\tau} \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_2 d\lambda_1, & -T < \tau < 0 \end{cases}$$

$$= T - |\tau| \quad \text{Therefore, } R_Y(\tau) = \frac{N_0}{2T} \left(1 - \frac{|\tau|}{T}\right)$$

$$\text{And, } \bar{Y}^2 = R_Y(\tau=0) = \frac{N_0}{2T}$$



Finite Time Average of White Noise

With zero mean white noise,  $R_X(\tau) = \frac{N_0}{2} \delta(\tau)$

$$R_Y(\tau) = \int_0^{\infty} d\lambda_1 \int_0^{\infty} R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2 = \frac{N_0}{2T^2} \int_0^T d\lambda_1 \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_2$$

$$\int_0^T \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_1 d\lambda_2 = T - |\tau| \quad \implies \quad R_Y(\tau) = \frac{N_0}{2T} \left(1 - \frac{|\tau|}{T}\right)$$

What is the corresponding transfer function?

**Example 8-6.1** White noise having a two-sided spectral density of 0.80 is applied to the input of a finite-time integrator having an impulse response of  $h(t) = \frac{1}{4}[u(t) - u(t-4)]$

Find the value of the autocorrelation function of the output at

a)  $\tau = 0$ ; b)  $\tau = 1$ ; c)  $\tau = 2$ .

$$T = 4 \text{ and } (N_0/2) = 0.8 \quad R_Y(\tau) = \frac{0.8}{4} \left(1 - \frac{|\tau|}{4}\right) = \frac{1}{5} \left(1 - \frac{|\tau|}{4}\right)$$

$$R_Y(0) = \frac{1}{5} \left(1 - \frac{0}{4}\right) = 0.2 \quad R_Y(1) = \frac{1}{5} \left(1 - \frac{1}{4}\right) = \frac{3}{20} = 0.15 \quad R_Y(2) = \frac{1}{5} \left(1 - \frac{2}{4}\right) = 0.1$$

Analysis of "Signal Plus Noise"

Consider a signal/process  $X(t) = A + V(t)$  where the noise  $V(t)$  has an autocorrelation function of  $R_V(\tau) = 10e^{-1000|\tau|}$ .

An RC circuit is being used to filter out the noise and the requirement is to measure  $A$  with an error of 1% when  $A$  is on the order of 1. Determine the RC time constant.

For the noise:  $R_V(\tau) = \frac{\beta S_0}{2} e^{-\beta|\tau|} = 10e^{-1000|\tau|} \quad \beta = 1000, S_0 = 0.02$   
A wideband noise

$$S_V(\omega) = \int_{-\infty}^{\infty} R_V(\tau) e^{-j\omega\tau} d\tau \text{ and } S_V(0) = \int_{-\infty}^{\infty} R_V(\tau) d\tau = 2 \int_0^{\infty} 10e^{-1000\tau} d\tau = 0.02$$

around  $\omega = 0$ ,  $S_V(\omega) \approx S_V(0) = 0.02$

$Y(t) = A + U(t)$  where  $U(t)$  is the filtered version of  $V(t)$

$$\bar{U}^2 = R_U(0) \approx \frac{bN_0}{4} = \frac{bS_V(0)}{2} = 0.01b, \text{ or } \sqrt{U^2} \approx 0.1\sqrt{b}$$

The requirement dictates that  $\sqrt{U^2} \approx 0.1\sqrt{b} \leq 1\% = 0.01 \implies b \leq 0.01$

Since  $b = 1/RC \implies RC \geq 100$

## Example 8-6.2

Consider a process of a dc signal plus noise  $X(t) = A + V(t)$  where  $V(t)$  has an autocorrelation function of  $R_V(\tau) = 1 - \frac{|\tau|}{0.02}$  for  $|\tau| \leq 0.02$ . A finite time integrator is used to estimate the value of  $A$  with the expectation that the rms error is less than 0.01. If the impulse response of the integrator is  $h(t) = \frac{1}{T}[u(t) - u(t - T)]$  find the value of  $T$  to accomplish this.

$$S_V(\omega) = \int_{-\infty}^{\infty} R_V(\tau) d\tau = 2 \int_0^{0.02} \left(1 - \frac{\tau}{0.02}\right) d\tau = 2 \times 0.01 = 0.02$$

around  $\omega = 0$ ,  $S_V(\omega) \approx S_V(0) = 0.02$

$Y(t) = A + U(t)$  where  $U(t)$  is the filtered version of  $V(t)$

Since the signal is a constant and the expected error is small, it is reasonable to assume that within the passband, the noise power is constant (equivalent to white noise).

$$R_U(\tau) = \frac{S_V(0)}{T} \left(1 - \frac{|\tau|}{T}\right) \text{ and } \overline{U^2} = R_U(0) = \frac{S_V(0)}{T} = \frac{0.02}{T}$$

We require that  $\overline{U^2} = \frac{0.02}{T} \leq (0.01)^2$ . Therefore,  $T \geq 200$ .

## Frequency Domain Analysis

As discussed previously, to avoid technical difficulties associated with the existence of Fourier transform of random processes, we focus on the method of spectral density.

$$S_X(\omega) = \mathbf{F}\{R_X(\tau)\}$$

$$R_Y(\tau) = \int_0^{\infty} d\lambda_1 \int_0^{\infty} R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2$$

$$S_Y(\omega) = \mathbf{F}\{R_Y(\tau)\} = \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} d\lambda_1 \int_0^{\infty} R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2 \right\} e^{-j\omega\tau} d\tau$$

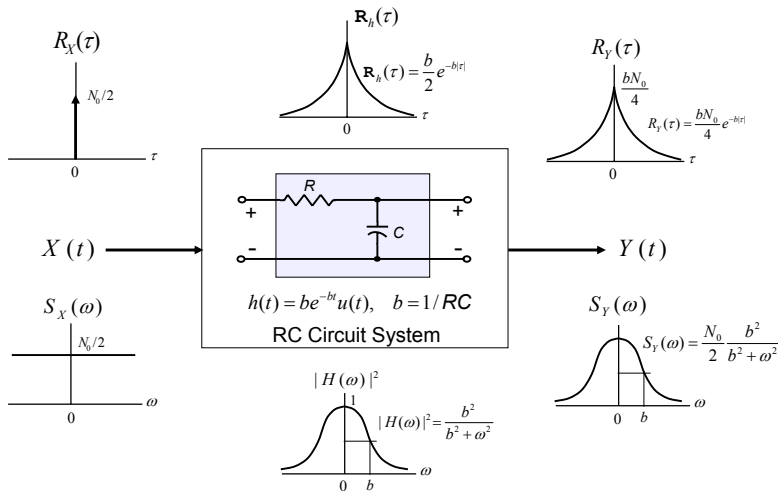
$$S_Y(\omega) = \int_0^{\infty} d\lambda_1 \int_0^{\infty} h(\lambda_1) h(\lambda_2) d\lambda_2 \int_{-\infty}^{\infty} R_X(\lambda_2 - \lambda_1 - \tau) e^{-j\omega\tau} d\tau$$

$$= \int_0^{\infty} d\lambda_1 \int_0^{\infty} h(\lambda_1) h(\lambda_2) S_X(\omega) e^{-j\omega(\lambda_2 - \lambda_1)} d\lambda_2$$

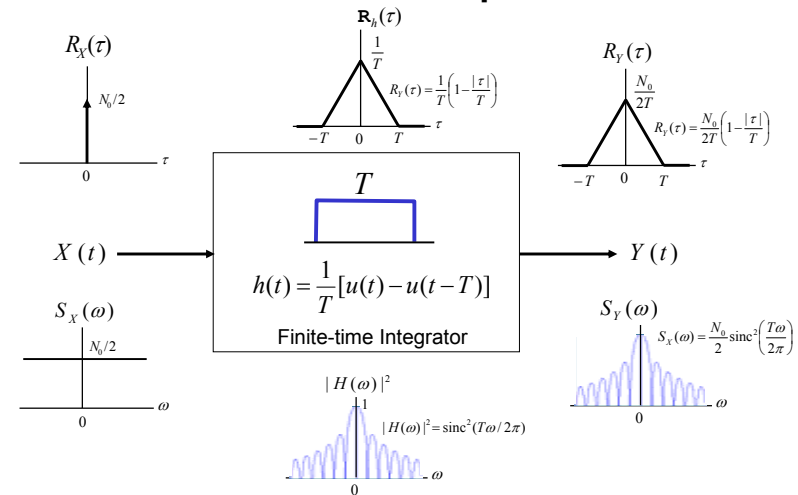
$$= S_X(\omega) \int_0^{\infty} h(\lambda_1) e^{j\omega\lambda_1} d\lambda_1 \int_0^{\infty} h(\lambda_2) e^{-j\omega\lambda_2} d\lambda_2$$

$$= S_X(\omega) H(-\omega) H(\omega) = S_X(\omega) |H(\omega)|^2$$

## RC Circuit – Input Output Relationship



## Finite-Time Integrator – Input Output Relationship



## Example

White noise having two-sided spectral density of  $1 \text{ V}^2/\text{Hz}$  is applied to the input of a linear system having an impulse response of  $h(t) = te^{-2t}u(t)$

1. Find the value of the output spectral density at  $\omega = 0$ .
2. Find the value of the output spectral density at  $\omega = 3$ .
3. Find the mean-square value of the output.

$$\mathbf{F}\{e^{-2t}u(t)\} = \frac{1}{2+j\omega} \quad \mathbf{F}\{-jh(t)\} = \mathbf{F}\{-jte^{-2t}u(t)\} = \frac{d}{d\omega} \frac{1}{2+j\omega} = \frac{-j}{(2+j\omega)^2}$$

$$\Rightarrow \mathbf{F}\{h(t)\} = \mathbf{F}\{te^{-2t}u(t)\} = \frac{1}{(2+j\omega)^2} = H(\omega)$$

$$|H(\omega)|^2 = \frac{1}{(4+\omega^2)^2} \text{ and } S_y(\omega) = S_x(\omega) |H(\omega)|^2 = \frac{1}{(4+\omega^2)^2}$$

$$S_y(0) = \frac{1}{(4+0^2)^2} = \frac{1}{16} = 0.0625 \quad S_y(1) = \frac{1}{(4+1^2)^2} = \frac{1}{25} = 0.04 \quad S_y(3) = \frac{1}{(4+3^2)^2} = \frac{1}{169} = 0.0059$$

$$\overline{Y^2} = \int_0^\infty h^2(t) dt = \int_0^\infty t^2 e^{-4t} dt = \frac{e^{-4t} (16t^2 + 8t + 2)}{-64} \Big|_0^\infty = \frac{2}{64} = 0.03125$$

## Example – Cont'd

In the previous example, the mean-square value can also be calculated through integration of the spectral density over the entire frequency range:

$$\begin{aligned} \overline{Y^2} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(4+\omega^2)^2} d\omega \\ &= \frac{1}{2\pi} \left[ \frac{\omega}{8(4+\omega^2)} + \frac{1}{16} \tan^{-1}(\omega/2) \right]_{-\infty}^{\infty} = \frac{1}{2\pi} \frac{1}{16} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{1}{32} = 0.03125 \end{aligned}$$

The rms bandwidth of the system:

$$\begin{aligned} \int_0^\infty |H(\omega)|^2 d\omega &= \int_0^\infty \frac{1}{(4+\omega^2)^2} d\omega = \left[ \frac{\omega}{8(4+\omega^2)} + \frac{1}{16} \tan^{-1}(\omega/2) \right]_0^\infty = \frac{1}{16} \left( \frac{\pi}{2} \right) = \frac{\pi}{32} \\ B_{rms}^2 &= \frac{\int_0^\infty \omega^2 |H(\omega)|^2 d\omega}{\int_0^\infty |H(\omega)|^2 d\omega} = \frac{32}{\pi} \int_0^\infty \frac{\omega^2}{(4+\omega^2)^2} d\omega \\ &= \frac{32}{\pi} \left[ \frac{-\omega}{2(4+\omega^2)} + \frac{1}{4} \tan^{-1}(\omega/2) \right]_0^\infty = \frac{32}{\pi} \frac{\pi}{8} = 4 \Rightarrow B_{rms} = 2 \end{aligned}$$

## Example

A linear system has an impulse response of  $h(t) = te^{-2t}u(t)$ . If the input has a spectral density of  $S_x(\omega) = \frac{1800}{900+\omega^2}$

1. Find the value of the output spectral density at  $\omega = 0$ .
2. Find the mean-square value of the output.

$$|H(\omega)|^2 = \frac{1}{(4+\omega^2)^2} \text{ and } S_y(\omega) = S_x(\omega) |H(\omega)|^2 = \frac{1800}{900+\omega^2} \frac{1}{(4+\omega^2)^2}$$

$$S_y(0) = \frac{1800}{900+0^2} \frac{1}{(4+0^2)^2} = \frac{1}{8} = 0.125$$

$$S_x(\omega) = \frac{2\beta A}{\beta^2 + \omega^2} \iff R_x(\tau) = Ae^{-\beta|\tau|} \quad A > 0, \beta > 0 \Rightarrow \beta = 30 \gg 2, A = 30$$

$$S_x(\omega) = \frac{1800}{900 + \omega^2} \iff R_x(\tau) = 30e^{-30|\tau|}$$

We consider the input bandwidth to be much greater than the system bandwidth, and therefore

$$\overline{Y^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1800}{900 + \omega^2} \frac{1}{(4 + \omega^2)^2} d\omega \approx \frac{2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(4 + \omega^2)^2} d\omega = 0.062$$

## Equivalent Baseband Noise Bandwidth

The equivalent-noise bandwidth,  $B_{EN}$ , of a system is defined to be the bandwidth of an ideal filter that has the same maximum gain and the same mean-square value at its output as the actual system when the input is white noise.

$$B_{EN} = \frac{1}{2|H(0)|^2} \int_{-\infty}^{\infty} |H(f)|^2 df = \frac{1}{4\pi|H(0)|^2} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

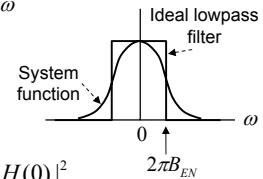
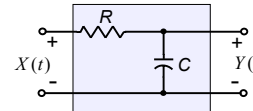
$$\text{Or, } 2|H(0)|^2 (2\pi B_{EN}) = \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

$$\text{If } S_x(\omega) = N_0/2, \quad S_y(\omega) = \frac{N_0}{2} |H(\omega)|^2$$

$$\overline{Y^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) d\omega = \frac{N_0}{4\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega = N_0 B_{EN} |H(0)|^2$$

$$\overline{Y^2} = R_y(\tau=0) = \frac{bN_0}{4} e^{-b|0|} = \frac{bN_0}{4}$$

$$N_0 B_{EN} |H(0)|^2 = N_0 B_{EN} = \frac{bN_0}{4} \quad B_{EN} = \frac{b}{4} = \frac{1}{4RC}$$



The ENB of an RC circuit is  $\frac{1}{4RC}$  (Hz) or  $\frac{\pi}{2RC}$  (rad/s)

## Half-Power BW & ENB of RC Circuit

Recall  $\mathbf{F}\{h(t)\} = H(\omega) = \frac{b}{b + j\omega}$  and  $|H(\omega)|^2 = \frac{b}{b + j\omega} \frac{b}{b - j\omega} = \frac{b^2}{b^2 + \omega^2}$

Half - Power (or 3 - dB) Bandwidth ( for lowpass signal) is the frequency

at which  $|H(\omega = 2\pi B_{1/2})|^2 = \frac{1}{2} |H(0)|^2$

For RC circuits,  $|H(\omega = b)|^2 = \frac{b^2}{b^2 + b^2} = \frac{1}{2} = \frac{1}{2} |H(0)|^2$

Therefore,  $2\pi B_{1/2} = b = 4B_{EN}$  or  $B_{EN} = \frac{\pi}{2} B_{1/2} = 1.57 B_{1/2}$  for a RC circuit.

Expressed in time domain,

$$H(0) = \int_0^\infty h(t) dt \quad \int_0^\infty h^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^\infty |H(\omega)|^2 d\omega$$

$$2\pi B_{EN} = \frac{\int_{-\infty}^\infty |H(\omega)|^2 d\omega}{2 |H(0)|^2} = \frac{2\pi \int_0^\infty h^2(t) dt}{2 \left[ \int_0^\infty h(t) dt \right]^2} = \frac{\pi \int_0^\infty h^2(t) dt}{\left[ \int_0^\infty h(t) dt \right]^2}$$

Time-domain representation has advantage when the system transfer function is non-rational.

## Bandwidth of A Finite-Time Integrator

Consider a finite-time integrator:  $h(t) = \frac{1}{T} [u(t) - u(t - T)]$

$$\int_0^\infty h(t) dt = \frac{1}{T} T = 1 \quad \int_0^\infty h^2(t) dt = \frac{1}{T^2} T = \frac{1}{T}$$

$$2\pi B_{EN} = \frac{\int_{-\infty}^\infty |H(\omega)|^2 d\omega}{2 |H(0)|^2} = \frac{\pi \int_0^\infty h^2(t) dt}{\left[ \int_0^\infty h(t) dt \right]^2} = \frac{\pi}{T} \quad \text{or } B_{EN} = \frac{1}{2T}$$

$$H(\omega) = \mathbf{F}\{h(t)\} = \mathbf{F}\left\{ \frac{1}{T} \text{Rect}\left(\frac{t}{T} - \frac{1}{2}\right) \right\} = \frac{1}{T} \mathbf{F}\left\{ \text{Rect}\left(\frac{t - (T/2)}{T}\right) \right\}$$

$$= \frac{1}{T} \mathbf{F}\left\{ \text{Rect}\left(\frac{t}{T}\right) \right\} e^{-j\omega T/2} = \frac{1}{T} T \text{sinc}\left(\frac{\omega T}{2\pi}\right) e^{-j\omega T/2} = \text{sinc}\left(\frac{\omega T}{2\pi}\right) e^{-j\omega T/2}$$

$$\Rightarrow |H(\omega)|^2 = \text{sinc}^2\left(\frac{\omega T}{2\pi}\right)$$

$$|H(2\pi B_{1/2})|^2 = \text{sinc}^2(B_{1/2} T) = \frac{1}{2}, \quad B_{1/2} T = 0.44,$$

$$\text{Therefore, } B_{1/2} = \frac{0.44}{T} \cong \frac{0.9 \times 0.5}{T} = 0.9 B_{EN}$$

## Time Derivative

Let  $\dot{X}(t) = dX(t) / dt$

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^\infty F_X(\omega) e^{j\omega t} d\omega$$

$$\dot{X}(t) = \frac{dX(t)}{dt} = \frac{d}{dt} \frac{1}{2\pi} \int_{-\infty}^\infty F_X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^\infty j\omega F_X(\omega) e^{j\omega t} d\omega$$

$$\mathbf{F}\{\dot{X}(t)\} = j\omega F_X(\omega) \quad S_{\dot{X}}(\omega) = \omega^2 \lim_{T \rightarrow \infty} \frac{E[|F_{X_T}(\omega)|^2]}{2T} = \omega^2 S_X(\omega)$$

Consider a signal/process  $X(t) = A + V(t)$  where A is a random variable uniformly distributed in (4,6) and the noise  $V(t)$  has an autocorrelation function  $R_V(\tau) = 3\delta(\tau)$ . Find the power of the time derivative of the signal with frequency within 10 rad/s.

$$R_X(\tau) = A^2 + 3\delta(\tau) = 25.33 + 3\delta(\tau)$$

$$S_X(\omega) = \mathbf{F}\{R_X(\tau)\} = 25.3 \times 2\pi\delta(\omega) + 3 = 159.2\delta(\omega) + 3$$

$$S_{\dot{X}}(\omega) = \omega^2 S_X(\omega) = 159.2\delta(\omega)\omega^2 + 3\omega^2$$

$$\int_{-10}^{10} S_{\dot{X}}(\omega) d\omega = \int_{-10}^{10} [159.2\delta(\omega)\omega^2 + 3\omega^2] d\omega = \omega^3 \Big|_{-10}^{10} = 2000$$