ECE 3075A
Random Signals

Lecture 23
System Analysis in Time & Frequency Domain

School of Electrical and Computer Engineering
Georgia Institute of Technology
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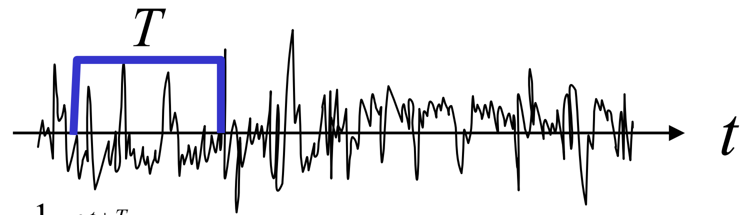
Finite Time Average of White Noise

The input is zero mean white noise, $R_X(\tau)=\frac{N_0}{2}\delta(\tau)$



$\frac{1}{T}\int_t^{t+T}(\bullet)\,d\lambda$

$$\overline{Y}=\overline{X}\int_0^T\frac{1}{T}dt=\overline{X}=0$$

$$R_Y(\tau)=\int_0^{\infty}d\lambda_1\int_0^{\infty}R_X(\lambda_2-\lambda_1-\tau)h(\lambda_1)h(\lambda_2)d\lambda_2$$

$$=\frac{N_0}{2}\int_0^T d\lambda_1\int_0^T\delta(\lambda_2-\lambda_1-\tau)\frac{1}{T}\frac{1}{T}d\lambda_2$$

$$\int_0^T\int_0^T\delta(\lambda_2-\lambda_1-\tau)d\lambda_1 d\lambda_2$$

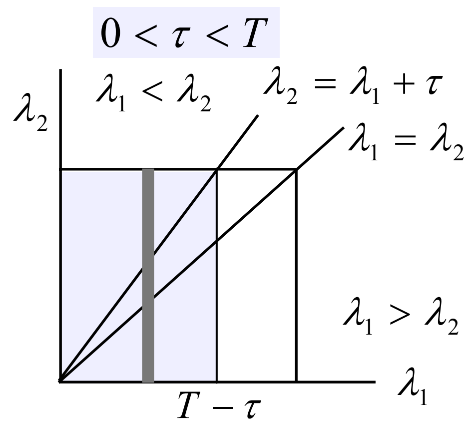
$$=\begin{cases}\int_0^{T-\tau}\int_0^T\delta(\lambda_2-\lambda_1-\tau)d\lambda_2 d\lambda_1, & 0<\tau<T\\ \int_0^{T+\tau}\int_0^T\delta(\lambda_2-\lambda_1-\tau)d\lambda_1 d\lambda_2, & -T<\tau<0\end{cases}$$

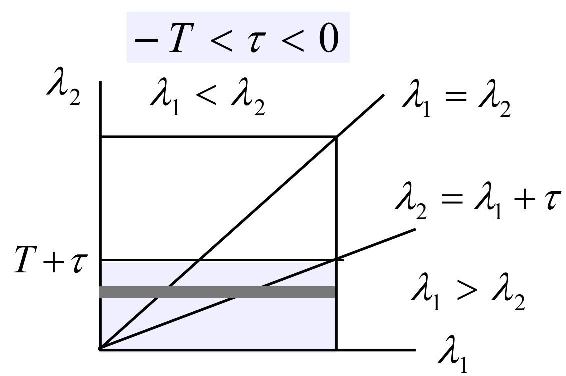
$$=T-|\tau|$$

$|\tau|<T$

Therefore, $R_Y(\tau)=\frac{N_0}{2T}\left(1-\frac{|\tau|}{T}\right)$

And, $\overline{Y^2}=R_Y(\tau=0)=\frac{N_0}{2T}$





Finite Time Average of White Noise

With zero mean white noise, $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$

$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2 = \frac{N_0}{2T^2}\int_0^T d\lambda_1 \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_2$$

$$\int_0^T \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_1 d\lambda_2 = \underset{|\tau| < T}{T - |\tau|} \quad \Rightarrow \quad R_Y(\tau) = \frac{N_0}{2T}\left(1 - \frac{|\tau|}{T}\right)$$

What is the corresponding transfer function?

Example 8-6.1 White noise having a two-sided spectral density of 0.80 is applied to the input of a finite-time integrator having an impulse response of

$$h(t) = \frac{1}{4}[u(t) - u(t-4)]$$

Find the value of the autocorrelation function of the output at

a) $\tau = 0$; b) $\tau = 1$; c) $\tau = 2$.

$T = 4$ and $(N_0/2) = 0.8$ $\quad R_Y(\tau) = \frac{0.8}{4}\left(1 - \frac{|\tau|}{4}\right) = \frac{1}{5}\left(1 - \frac{|\tau|}{4}\right)$

$R_Y(0) = \frac{1}{5}\left(1 - \frac{|0|}{4}\right) = 0.2$ $\quad R_Y(1) = \frac{1}{5}\left(1 - \frac{1}{4}\right) = \frac{3}{20} = 0.15$ $\quad R_Y(2) = \frac{1}{5}\left(1 - \frac{2}{4}\right) = 0.1$

Analysis of "Signal Plus Noise"

Consider a signal/process $X(t) = A + V(t)$ where the noise $V(t)$ has an autocorrelation function of $R_V(\tau) = 10e^{-1000|\tau|}$.
An RC circuit is being used to filter out the noise and the requirement is to measure A with an error of 1% when A is on the order of 1. Determine the RC time constant.

For the noise: $R_V(\tau) = \frac{\beta S_0}{2} e^{-\beta|\tau|} = 10e^{-1000|\tau|}$ $\beta = 1000, S_0 = 0.02$
A wideband noise

$S_V(\omega) = \int_{-\infty}^{\infty} R_V(\tau) e^{-j\omega\tau} d\tau$ and $S_V(0) = \int_{-\infty}^{\infty} R_V(\tau) d\tau = 2\int_0^{\infty} 10e^{-1000\tau} d\tau = 0.02$

around $\omega = 0$, $S_V(\omega) \approx S_V(0) = 0.02$

$Y(t) = A + U(t)$ where $U(t)$ is the filtered version of $V(t)$

$$\overline{U^2} = R_U(0) \approx \frac{bN_0}{4} = \frac{bS_V(0)}{2} = 0.01b, \quad \text{or} \quad \sqrt{\overline{U^2}} \approx 0.1\sqrt{b}$$

The requirement dictates that $\sqrt{\overline{U^2}} \approx 0.1\sqrt{b} \le 1\% = 0.01 \quad \Rightarrow \quad b \le 0.01$

Since $b = 1/RC \quad \Rightarrow \quad RC \ge 100$

Example 8-6.2

Consider a process of a dc signal plus noise $X(t) = A + V(t)$ where $V(t)$ has an autocorrelation function of $R_V(\tau) = 1 - \frac{|\tau|}{0.02}$ for $|\tau| \le 0.02.$ A finite time integrator is used to estimate the value of A with the expectation that the rms error is less than 0.01. If the impulse response of the integrator is $h(t) = \frac{1}{T}[u(t) - u(t-T)]$ find the value of T to accomplish this.

$$S_V(0) = \int_{-\infty}^{\infty} R_V(\tau)d\tau = 2\int_0^{0.02}\left(1 - \frac{\tau}{0.02}\right)d\tau = 2\times 0.01 = 0.02$$

around $\omega = 0$, $S_V(\omega) \approx S_V(0) = 0.02$

$Y(t) = A + U(t)$ where $U(t)$ is the filtered version of $V(t)$

Since the signal is a constant and the expected error is small, it is reasonable to assume that within the passband, the noise power is constant (equivalent to white noise).

$$R_U(\tau) = \frac{S_V(0)}{T}\left(1 - \frac{|\tau|}{T}\right) \text{ and } \overline{U^2} = R_U(0) = \frac{S_V(0)}{T} = \frac{0.02}{T}$$

We require that $\overline{U^2} = \frac{0.02}{T} \le (0.01)^2.$ Therefore, $T \ge 200.$

Frequency Domain Analysis

As discussed previously, to avoid technical difficulties associated with the existence of Fourier transform of random processes, we focus on the method of spectral density.

$$S_X(\omega) = \mathbf{F}\{R_X(\tau)\}$$

$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2$$

$$S_Y(\omega) = \mathbf{F}\{R_Y(\tau)\} = \int_{-\infty}^\infty \left\{ \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2 \right\} e^{-j\omega\tau} d\tau$$

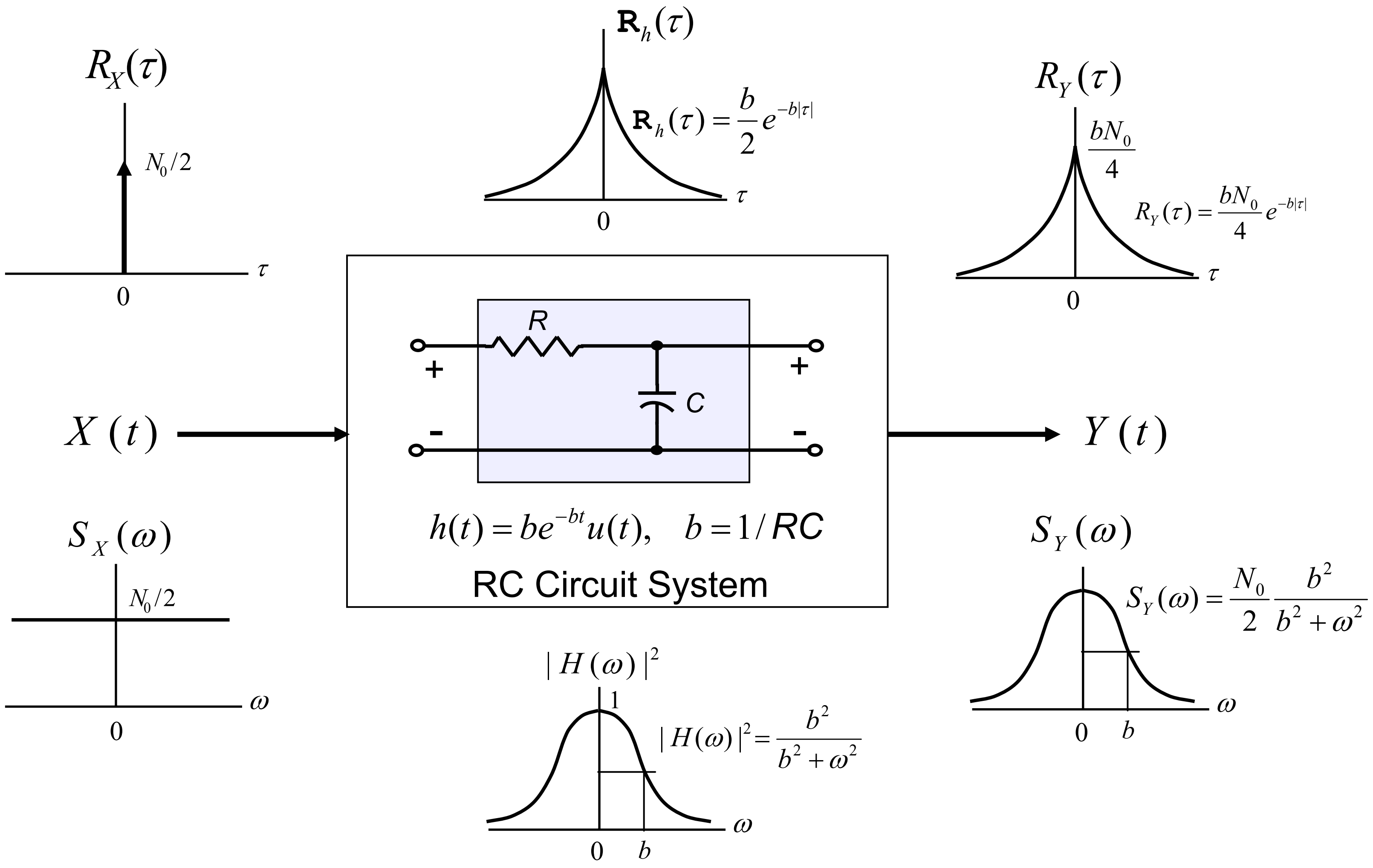
$$S_Y(\omega) = \int_0^\infty d\lambda_1 \int_0^\infty h(\lambda_1) h(\lambda_2) d\lambda_2 \int_{-\infty}^\infty R_X(\lambda_2 - \lambda_1 - \tau) e^{-j\omega\tau} d\tau$$

$$= \int_0^\infty d\lambda_1 \int_0^\infty h(\lambda_1) h(\lambda_2) S_X(\omega) e^{-j\omega(\lambda_2 - \lambda_1)} d\lambda_2$$

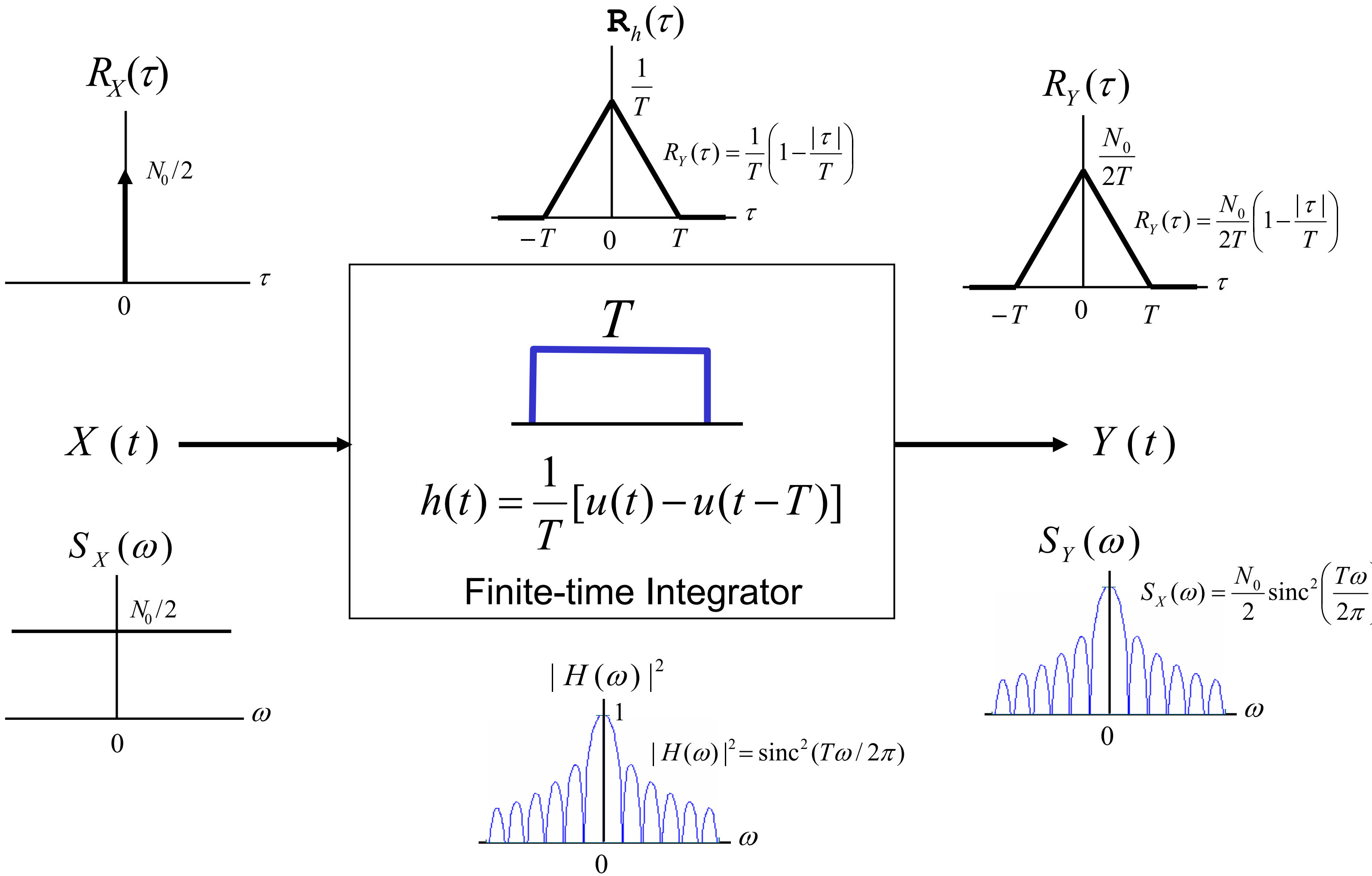
$$= S_X(\omega) \int_0^\infty h(\lambda_1) e^{j\omega\lambda_1)} d\lambda_1 \int_0^\infty h(\lambda_2) e^{-j\omega\lambda_2} d\lambda_2$$

$$= S_X(\omega) H(-\omega) H(\omega) = S_X(\omega) | H(\omega) |^2$$

RC Circuit – Input Output Relationship



Finite-Time Integrator – Input Output Relationship



Example

White noise having two-sided spectral density of 1 V^2/Hz is applied to the input of a linear system having an impulse response of $h(t) = te^{-2t}u(t)$

1. Find the value of the output spectral density at $\omega = 0$.
2. Find the value of the output spectral density at $\omega = 3$.
3. Find the mean-square value of the output.

$$\mathbf{F}\{e^{-2t}u(t)\} = \frac{1}{2+j\omega} \qquad \mathbf{F}\{-jh(t)\} = \mathbf{F}\{-jte^{-2t}u(t)\} = \frac{d}{d\omega}\frac{1}{2+j\omega} = \frac{-j}{(2+j\omega)^2}$$

$$\Rightarrow \quad \mathbf{F}\{h(t)\} = \mathbf{F}\{te^{-2t}u(t)\} = \frac{1}{(2+j\omega)^2} = H(\omega)$$

$|H(\omega)|^2 = \frac{1}{(4+\omega^2)^2}$ and $S_Y(\omega) = S_X(\omega)|H(\omega)|^2 = \frac{1}{(4+\omega^2)^2}$

$$S_Y(0) = \frac{1}{(4+0^2)^2} = \frac{1}{16} = 0.0625 \qquad S_Y(1) = \frac{1}{(4+1^2)^2} = \frac{1}{25} = 0.04 \qquad S_Y(3) = \frac{1}{(4+3^2)^2} = \frac{1}{169} = 0.0059$$

$$\overline{Y^2} = \int_0^\infty h^2(t)dt = \int_0^\infty t^2 e^{-4t}dt = \frac{e^{-4t}}{-64}(16t^2+8t+2)\Bigg|_0^\infty = \frac{2}{64} = 0.03125$$

Example – Cont'd

In the previous example, the mean-square value can also be calculated through integration of the spectral density over the entire frequency range:

$$\overline{Y^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(4+\omega^2)^2}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{\omega}{8(4+\omega^2)} + \frac{1}{16}\tan^{-1}(\omega/2)\Bigg|_{-\infty}^{\infty}\right] = \frac{1}{2\pi}\frac{1}{16}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{1}{32} = 0.03125$$

The rms bandwidth of the system:

$$\int_0^{\infty} |H(\omega)|^2\, d\omega = \int_0^{\infty} \frac{1}{(4+\omega^2)^2}d\omega = \left[\frac{\omega}{8(4+\omega^2)} + \frac{1}{16}\tan^{-1}(\omega/2)\Bigg|_0^{\infty}\right] = \frac{1}{16}\left(\frac{\pi}{2}\right) = \frac{\pi}{32}$$

$$B_{rms}^2 = \frac{\int_0^{\infty} \omega^2 |H(\omega)|^2\, d\omega}{\int_0^{\infty} |H(\omega)|^2\, d\omega} = \frac{32}{\pi}\int_0^{\infty} \frac{\omega^2}{(4+\omega^2)^2}d\omega$$

$$= \frac{32}{\pi}\left[\frac{-\omega}{2(4+\omega^2)} + \frac{1}{4}\tan^{-1}(\omega/2)\right]\Bigg|_0^{\infty} = \frac{32}{\pi}\frac{\pi}{8} = 4 \quad \Rightarrow \quad B_{rms} = 2$$

Example

A linear system has an impulse response of $h(t) = te^{-2t}u(t)$.
If the input has a spectral density of $S_X(\omega) = \dfrac{1800}{900+\omega^2}$

1. Find the value of the output spectral density at $\omega = 0$.
2. Find the mean-square value of the output.

$|H(\omega)|^2 = \dfrac{1}{(4+\omega^2)^2}$ and $S_Y(\omega) = S_X(\omega)|H(\omega)|^2 = \dfrac{1800}{900+\omega^2}\dfrac{1}{(4+\omega^2)^2}$

$$S_Y(0) = \frac{1800}{900+0^2}\frac{1}{(4+0^2)^2} = \frac{1}{8} = 0.125$$

$$S_X(\omega) = \frac{2\beta A}{\beta^2+\omega^2} \Longleftrightarrow R_X(\tau) = Ae^{-\beta|\tau|} \quad A>0,\ \beta>0 \quad \Rightarrow \quad \beta = 30 >> 2,\ A = 30$$

$$S_X(\omega) = \frac{1800}{900+\omega^2} \Longleftrightarrow R_X(\tau) = 30e^{-30|\tau|}$$

We consider the input bandwidth to be much greater than the system bandwidth, and therefore

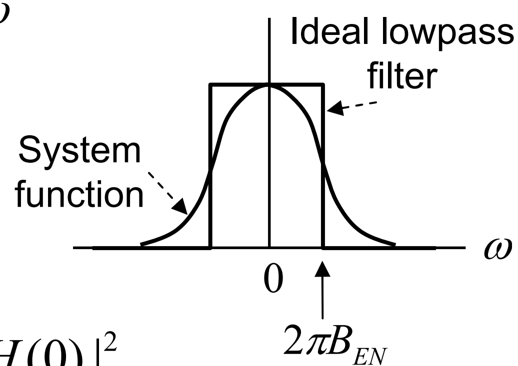
$$\overline{Y^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1800}{900+\omega^2}\frac{1}{(4+\omega^2)^2}d\omega \approx \frac{2}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(4+\omega^2)^2}d\omega = 0.062$$

Equivalent Baseband Noise Bandwidth

The equivalent-noise bandwidth, B_{EN}, of a system is defined to be the bandwidth of an ideal filter that has the same maximum gain and the same mean-square value at its output as the actual system when the input is white noise.

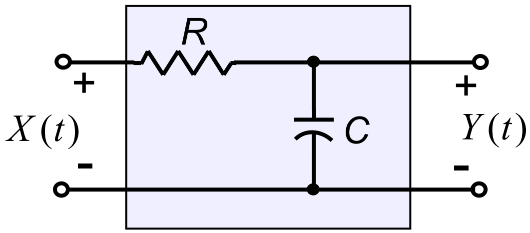
$$B_{EN} = \frac{1}{2|H(0)|^2}\int_{-\infty}^{\infty}|H(f)|^2\,df = \frac{1}{4\pi|H(0)|^2}\int_{-\infty}^{\infty}|H(\omega)|^2\,d\omega$$

Or, $2|H(0)|^2(2\pi B_{EN}) = \int_{-\infty}^{\infty}|H(\omega)|^2\,d\omega$



If $S_X(\omega) = N_0/2,\quad S_Y(\omega) = \frac{N_0}{2}|H(\omega)|^2$

$$\overline{Y^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}S_Y(\omega)d\omega = \frac{N_0}{4\pi}\int_{-\infty}^{\infty}|H(\omega)|^2\,d\omega = N_0 B_{EN}\,|H(0)|^2$$



$$\overline{Y^2} = R_Y(\tau = 0) = \frac{bN_0}{4}e^{-b|0|} = \frac{bN_0}{4}$$

$$N_0 B_{EN}\,|H(0)|^2 = N_0 B_{EN} = \frac{bN_0}{4} \qquad B_{EN} = \frac{b}{4} = \frac{1}{4RC}$$

The ENB of an RC circuit is $\frac{1}{4RC}$ (Hz) or $\frac{\pi}{2RC}$ (rad/s)

Half-Power BW & ENB of RC Circuit

Recall $\mathbf{F}\{h(t)\} = H(\omega) = \dfrac{b}{b + j\omega}$ and $|H(\omega)|^2 = \dfrac{b}{b + j\omega}\dfrac{b}{b - j\omega} = \dfrac{b^2}{b^2 + \omega^2}$

Half - Power (or 3 - dB) Bandwidth (for lowpass signal) is the frequency

at which $|H(\omega = 2\pi B_{1/2})|^2 = \dfrac{1}{2}|H(0)|^2$

For RC circuits, $|H(\omega = b)|^2 = \dfrac{b^2}{b^2 + b^2} = \dfrac{1}{2} = \dfrac{1}{2}|H(0)|^2$

Therefore, $2\pi B_{1/2} = b = 4B_{EN}$ or $B_{EN} = \dfrac{\pi}{2}B_{1/2} = 1.57B_{1/2}$ for a RC circuit.

Expressed in time domain,

$$H(0) = \int_0^\infty h(t)dt \qquad \int_0^\infty h^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^\infty |H(\omega)|^2\, d\omega$$

$$2\pi B_{EN} = \frac{\int_{-\infty}^\infty |H(\omega)|^2\, d\omega}{2|H(0)|^2} = \frac{2\pi\int_0^\infty h^2(t)dt}{2\left[\int_0^\infty h(t)dt\right]^2} = \frac{\pi\int_0^\infty h^2(t)dt}{\left[\int_0^\infty h(t)dt\right]^2}$$

Time-domain representation has advantage when the system transfer function is non-rational.

Bandwidth of A Finite-Time Integrator

Consider a finite-time integrator: $h(t) = \frac{1}{T}[u(t) - u(t-T)]$

$$\int_0^\infty h(t)dt = \frac{1}{T}T = 1 \qquad \int_0^\infty h^2(t)dt = \frac{1}{T^2}T = \frac{1}{T}$$

$$2\pi B_{EN} = \frac{\int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega}{2|H(0)|^2} = \frac{\pi \int_0^\infty h^2(t)dt}{\left[\int_0^\infty h(t)dt\right]^2} = \frac{\pi}{T}$$

or $B_{EN} = \frac{1}{2T}$

$$H(\omega) = F\{h(t)\} = F\left\{\frac{1}{T}\text{Rect}\left(\frac{t}{T} - \frac{1}{2}\right)\right\} = \frac{1}{T}F\left\{\text{Rect}\left(\frac{t-(T/2)}{T}\right)\right\}$$

$$= \frac{1}{T}F\left\{\text{Rect}\left(\frac{t}{T}\right)\right\}e^{-j\omega T/2} = \frac{1}{T}T\text{sinc}\left(\frac{\omega T}{2\pi}\right)e^{-j\omega T/2} = \text{sinc}\left(\frac{\omega T}{2\pi}\right)e^{-j\omega T/2}$$

⇨ $|H(\omega)|^2 = \text{sinc}^2\left(\frac{\omega T}{2\pi}\right)$

$$|H(2\pi B_{1/2})|^2 = \text{sinc}^2\left(B_{1/2}T\right) = \frac{1}{2}, \quad B_{1/2}T = 0.44,$$

Therefore, $B_{1/2} = \frac{0.44}{T} \cong \frac{0.9 \times 0.5}{T} = 0.9 B_{EN}$

Time Derivative

Let $\dot{X}(t) = dX(t)/dt$

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_X(\omega)e^{j\omega t}d\omega$$

$$\dot{X}(t) = \frac{dX(t)}{dt} = \frac{d}{dt}\frac{1}{2\pi}\int_{-\infty}^{\infty} F_X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F_X(\omega)e^{j\omega t}d\omega$$

$$\mathbf{F}[\dot{X}(t)] = j\omega F_X(\omega)$$

$$S_{\dot{X}}(\omega) = \omega^2 \lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T} = \omega^2 S_X(\omega)$$

Consider a signal/process $X(t) = A + V(t)$ where A is a random variable uniformly distributed in (4,6) and the noise $V(t)$ has an autocorrelation function $R_V(\tau) = 3\delta(\tau)$. Find the power of the time derivative of the signal with frequency within 10 rad/s.

$$R_X(\tau) = \overline{A^2} + 3\delta(\tau) = 25.33 + 3\delta(\tau)$$

$$S_X(\omega) = \mathbf{F}\{R_X(\tau)\} = 25.3\times 2\pi\delta(\omega) + 3 = 159.2\delta(\omega) + 3$$

$$S_{\dot{X}}(\omega) = \omega^2 S_X(\omega) = 159.2\delta(\omega)\omega^2 + 3\omega^2$$

$$\int_{-10}^{10} S_{\dot{X}}(\omega)d\omega = \int_{-10}^{10}[159.2\delta(\omega)\omega^2 + 3\omega^2]d\omega = \omega^3\Big|_{-10}^{10} = 2000$$