

ECE 3075A Random Signals

Lecture 3

Conditional Probability, Independence and Revisit to Combined Experiment

School of Electrical and Computer Engineering
Georgia Institute of Technology
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Lecture #3, Slide #1

Interpretation of DeMoivre-Laplace Theorem

- When the number of trials is large, i.e., $npq \gg 1$ the mean approaches np and the variance approaches $np(1-p) = npq$
- Law of large numbers – asymptotic normality
- Deviation from normal distribution is small if k is in the vicinity of one standard deviation, i.e. $|k - np| < \sim \sqrt{npq} = \sigma$

$$npq \gg 1 \text{ and } |k - np| < \sim \sqrt{npq}$$

$$p_n(k) = \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} e^{-\frac{(k - np)^2}{2npq}}$$

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Lecture #3, Slide #2

Combined Experiment Again

Clarification of notations

In die - throwing experiment, event $A = \{1, 3\} = \{1\} \cup \{3\}$ occurs if the outcome of the trial is either 1 or 3, and

$$\Pr(A) = \Pr\{1\} + \Pr\{3\} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

If we write $A = (1, 3)$, then A is (and must be) an elementary outcome of a combined trial, consisting of 2 die - throwings, because a single die - throwing trial will never produce both 1 and 3 at the same time as the outcome; and if the "two trials

are independent", $\Pr(A) = \Pr\{1\} \times \Pr\{3\} = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$ assuming that (1,3) and (3,1) are different events.

Event & Conditional Event

- $\Pr(A)$ denotes the probability of event A .
 - $\Pr(A|B)$ denotes the probability of event A , under the circumstance that event B has occurred; i.e., the probability of occurrence of an event conditioned by the occurrence of another event.
- let $A = \{1, 2, 3\}$ and $B = \{1, 4, 5, 6\}$ be two events associated with a die - throwing experiment.

➔ Given that B has occurred, meaning among the observation of the four outcomes – 1, 4, 5, or 6, what is the probability that the occurrence of A is also observed? This is equivalent to asking about the probability of 1 given that B has occurred.

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Lecture #3, Slide #1

Interpretation of DeMoivre-Laplace Theorem

Conditional Probability

- If event B is assumed to have non-zero probability, the conditional probability of A , given B , is

$$\Pr(A|B) = \frac{\Pr(A, B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)}, \quad \Pr(B) > 0$$

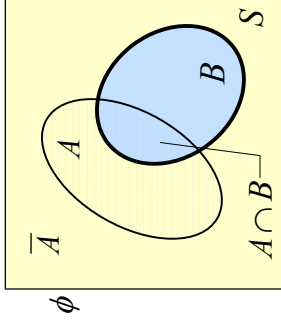
- We say joint event (A, B) occurs if the outcome of a trial satisfies the definition of both events, in other words, when any of the common elements of A and B appears as the outcome of the trial. That is,

$$\Pr(A, B) = \Pr(A \cap B)$$

In the previous example, $\Pr(A, B) = \Pr(A \cap B) = \Pr\{\} = \frac{1}{6}$

$$\text{Since } \Pr(B) = \frac{4}{6}, \quad \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/6}{4/6} = \frac{1}{4}$$

Venn Diagram



$$\Pr(B|B) = 1$$

$\Pr(A|B)$ can be considered as the probability of A when B is the observation space.

$$\Pr(A|B) = \frac{\Pr(A, B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)},$$

provided $\Pr(B) > 0$

Joint Probability

$\Pr(A, B) = \Pr(A \cap B) = \Pr(\text{both events } A \text{ and } B \text{ are observed})$

$$(A \cap B) \cup (\bar{A} \cap B) = S \cap B = B, \text{ and } (A \cap B) \cap (\bar{A} \cap B) = \phi \Rightarrow \Pr(A, B) + \Pr(\bar{A}, B) = \Pr(B)$$

In general, if A_i 's are mutually exclusive, $\bigcup_i A_i = S$, $\sum_{\text{all } i} \Pr(A_i, B) = \Pr(B)$

Some Obvious Results

- If two events A and B are mutually exclusive, $A \cap B = \phi$ and $\Pr(A \cap B) = 0 \Rightarrow \Pr(A|B) = 0$
- If one event contains another:

$$A \subset B \Rightarrow \Pr(A \cap B) = \Pr(A) \Rightarrow \Pr(A|B) = \frac{\Pr(A)}{\Pr(B)} \geq \Pr(A)$$

$$B \subset A \Rightarrow \Pr(A \cap B) = \Pr(B) \Rightarrow \Pr(A|B) = \frac{\Pr(B)}{\Pr(B)} = 1$$

- In general, nothing can be asserted regarding the magnitude of conditional probability.

Conditional Probability and Axioms

- Non-negativity: $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \geq 0$
- Sure event or total probability: $\Pr(S|B) = \frac{\Pr(S \cap B)}{\Pr(B)} = \frac{\Pr(B)}{\Pr(B)} = 1$
- Exclusivity/Additivity

Let C be another event such that $A \cap C = \phi$,

$$\Pr[(A \cup C) \cap B] = \Pr[(A \cap B) \cup (C \cap B)] = \Pr(A \cap B) + \Pr(C \cap B)$$

because $(A \cap B) \cap (C \cap B) = \phi$. Then

$$\Pr[(A \cup C) | B] = \frac{\Pr[(A \cup C) \cap B]}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)} + \frac{\Pr(C \cap B)}{\Pr(B)} = \Pr(A|B) + \Pr(C|B)$$

Example of Conditional Probability

Ohms	Bin Number						Total
	1	2	3	4	5	6	
10	500	0	200	800	1200	1000	3700
100	300	400	600	200	800	0	2300
1000	200	600	200	600	0	1000	2600
Total	1000	1000	1000	1600	2000	2000	8600

What is the probability of drawing a 10 ohm resistor from any bin?

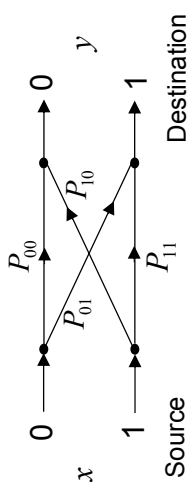
What is the probability of drawing a 100 ohm resistor from bin 3?

A 1000 ohm resistor was drawn. What is the probability that it came from bin 1?

Consider the experiment as one that produces (ohm, bin) as outcome.

Probability & Information Transmission

A binary channel



At the source,

$$\Pr(x=0) = p \text{ and}$$

$$\Pr(x=1) = 1 - p,$$

$$\Pr(x=0) \text{ and } \Pr(x=1)$$

are called **a priori** probabilities.

The channel, being non-ideal (causing confusions), is characterized by the four conditional probabilities: P_{00} , P_{01} , P_{10} , and P_{11}

$$P_{ij} = \Pr(y = j | x = i) = (\text{probability of } j \text{ received at the destination,}$$

i.e., $y = j$, when i was actually sent by the source, i.e., $x = i$)

A posteriori probability –

The probability that i was sent at

the source, given that j is received

at the destination.

$$\Pr(x = i | y = j) = \frac{\Pr(y = j | x = i) \Pr(x = i)}{\Pr(y = j)}$$

Bayes Formula

Is used to find a posteriori probability

$$\Pr(x = i | y = j) = \frac{\Pr(y = j | x = i) \Pr(x = i)}{\Pr(y = j)}$$

More precisely,

$\Pr(x = i | y = j) = \Pr(\text{symbol } i \text{ was sent, given that } j \text{ is received})$

$$= \frac{\Pr(y = j | x = i) \Pr(x = i)}{\sum_{all i} \Pr(y = j | x = i) \Pr(x = i)}$$

$$= \frac{P_{ij} \Pr(x = i)}{\sum_{all i} P_{ij} \Pr(x = i)} = \frac{\text{the source portion of contributions that led to the reception of } j}$$

Applications in Information Transmission

- When symbol 0 is sent, it has a probability of P_{01} being received as symbol 1 – i.e., error occurs.
- When symbol 1 is sent, it has a probability of P_{10} being received as symbol 0 – i.e., error occurs.
- What is the probability of error, regardless of the transmitted symbol?
 $pP_{01} + (1 - p)P_{10}$
- If we want to make sure of the delivery of information by repeating the same symbol n times, and our decision policy is to go with the majority of the n received symbols, what would be the probability of error in information reception?

Applications in Information Transmission

Repetitive transmission –
For every source symbol, we send it three times, i.e., $n = 3$

$$x=0 \quad P_{00} = P_{11} = 0.9 \\ P_{01} = P_{10} = 0.1$$

000	$P_{00}P_{00}P_{00}$	$y=0$	0.729
001	$P_{00}P_{00}P_{01}$	0	0.081
010	$P_{00}P_{01}P_{00}$	0	0.081
100	$P_{01}P_{00}P_{00}$	0	0.081
011	$P_{00}P_{01}P_{01}$	1	0.009
101	$P_{01}P_{00}P_{01}$	1	0.009
110	$P_{01}P_{01}P_{00}$	1	0.009
111	$P_{01}P_{01}P_{01}$	1	0.001

$\left. \begin{matrix} + \\ + \end{matrix} \right\} = 0.972$ Probability of correct reception
 $\left. \begin{matrix} + \\ + \end{matrix} \right\} = 0.028$ Probability of incorrect reception

The error probability is 0.028, compared to 0.1 with single transmission.

Efficiency of repetitive code is low, though. Other error correction code can have much higher performance.

- Recall combined experiments and combined trials
 - experiment 1 \Rightarrow {event A_1 }
 - experiment 2 \Rightarrow {event A_2 }
 - experiment $(1 \times 2) \Rightarrow$ {event $A_{1 \times 2}$ } \times {event A_2 }

Two events (of two trials), A_1 and A_2 , are independent if and only if $\Pr(A_1 \cap A_2) = \Pr(A_1) \Pr(A_2) \cap$ means "and"

- Extension to n ($n > 2$) events:

$$\Pr(A_1 \cap A_2 \cap \dots \cap A_n) = \Pr(A_1) \Pr(A_2) \dots \Pr(A_n)$$

for any set of integers $\leq n$. Total number of equations to establish independence is $2^n - (n + 1)$.

- Be careful not to get confused when A_1 and A_2 are events of the same experiment.

Independence

- If A and B are independent events of the same experiment, what can we say about them?

$$\Pr(A \cap B) = \Pr(A) \Pr(B)$$

$$\Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A) \Pr(B)}{\Pr(B)} = \Pr(A)$$

$$\Pr(B | A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{\Pr(A) \Pr(B)}{\Pr(A)} = \Pr(B)$$

That is, the probability assignment of event A has nothing to do with that of event B and vice versa.