ECE 3075A
Random Signals

Lecture 3
Conditional Probability, Independence and Revisit to Combined Experiment

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Interpretation of DeMoivre-Laplace Theorem

- When the number of trials is large, i.e., $npq >> 1$ the mean approaches np and the variance approaches $np(1\text{-}p)=npq$
- Law of large numbers – asymptotic normality
- Deviation from normal distribution is small if k is in the vicinity of one standard deviation, i.e. $|k\text{-}np| <\sim \sqrt{npq} = \sigma$

$npq >> 1$ and $|k\text{-}np| <\sim \sqrt{npq}$

$$p_n(k) = \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)^2/2npq}$$

Combined Experiment Again

Clarification of notations

In die - throwing experiment, event $A = \{1, 3\} = \{1\} \cup \{3\}$ occurs if the outcome of the trial is either 1 or 3, and

$$\Pr(A) = \Pr\{1\} + \Pr\{3\} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

If we write $A = (1, 3)$, then A is (and must be) an elementary outcome of a combined trial, consisting of 2 die - throwings, because a single die - throwing trial will never produce both 1 and 3 at the same time as the outcome; and if the "two trials are independent",

$$\Pr(A) = \Pr\{1\} \times \Pr\{3\} = \frac{1}{6} \bullet \frac{1}{6} = \frac{1}{36}$$

assuming that $(1,3)$ and $(3,1)$ are different events.

Event & Conditional Event

- $\Pr(A)$ denotes the probability of event A.
- $\Pr(A \mid B)$ denotes the probability of event A, under the circumstance that event B has occurred; i.e., the probability of occurrence of an event conditioned by the occurrence of another event.

let $A = \{1, 2, 3\}$ and $B = \{1, 4, 5, 6\}$ be two events associated with a die - throwing experiment.

➡ Given that B has occurred, meaning among the observation of the four outcomes – $1, 4, 5,$ or 6, what is the probability that the occurrence of A is also observed? This is equivalent to asking about the probability of 1 given that B has occurred.

Conditional Probability

- If event B is assumed to have non-zero probability, the conditional probability of A, given B, is

$$\Pr(A \mid B) = \frac{\Pr(A,B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)}, \qquad \Pr(B) > 0$$

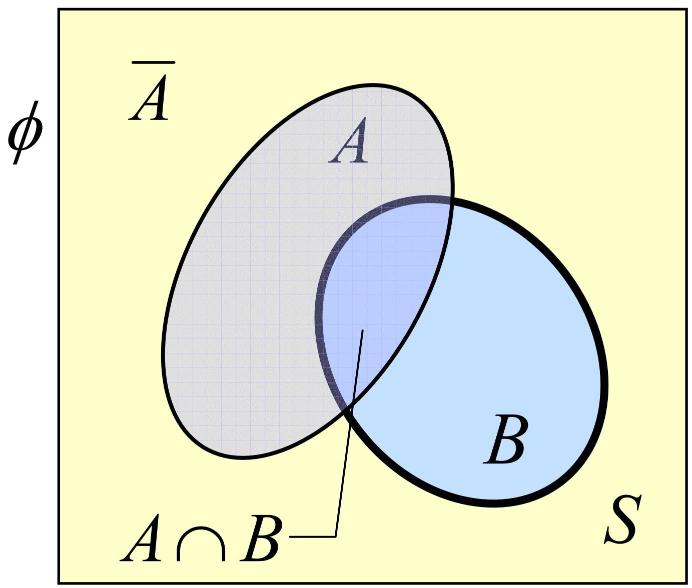
- We say joint event (A, B) occurs if the outcome of a trial satisfies the definition of both events, in other words, when any of the common elements of A and B appears as the outcome of the trial. That is,

$$\Pr(A, B) = \Pr(A \cap B)$$

In the previous example, $\Pr(A, B) = \Pr(A \cap B) = \Pr\{1\} = \frac{1}{6}$

Since $\Pr(B) = \frac{4}{6}$, $\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/6}{4/6} = \frac{1}{4}$

Venn Diagram



$\Pr(B \mid B) = 1$

$\Pr(A \mid B)$ can be considered as the probability of A when B is the observation space.

$$\Pr(A \mid B) = \frac{\Pr(A, B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)},$$

provided $\Pr(B) > 0$

Joint Probability

$\Pr(A, B) = \Pr(A \cap B) = \Pr($both events A and B are observed$)$

$(A \cap B) \cup (\overline{A} \cap B) = S \cap B = B$, and $(A \cap B) \cap (\overline{A} \cap B) = \phi \;\Rightarrow\; \Pr(A, B) + \Pr(\overline{A}, B) = \Pr(B)$

In general, if $A_i s$ **are mutually exclusive,** $\bigcup_i A_i = S$, $\sum_{All\ i} \Pr(A_i, B) = \Pr(B)$

Some Obvious Results

- If two events A and B are **mutually exclusive**,

  $A \cap B = \phi$ and $\Pr(A \cap B) = 0 \quad \Rightarrow \quad \Pr(A \mid B) = 0$

- If one event contains another:

$$A \subset B \;\Rightarrow\; \Pr(A \cap B) = \Pr(A) \;\Rightarrow\; \Pr(A \mid B) = \frac{\Pr(A)}{\Pr(B)} \ge \Pr(A)$$

$$B \subset A \;\Rightarrow\; \Pr(A \cap B) = \Pr(B) \;\Rightarrow\; \Pr(A \mid B) = \frac{\Pr(B)}{\Pr(B)} = 1$$

- In general, nothing can be asserted regarding the magnitude of conditional probability.

Conditional Probability and Axioms

- Non-negativity: $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} \geq 0$
- Sure event or total probability:

$$\Pr(S \mid B) = \frac{\Pr(S \cap B)}{\Pr(B)} = \frac{\Pr(B)}{\Pr(B)} = 1$$

- Exclusivity/Additivity

Let C be another event such that $A \cap C = \phi$,

$$\Pr[(A \cup C) \cap B] = \Pr[(A \cap B) \cup (C \cap B)] = \Pr(A \cap B) + \Pr(C \cap B)$$

because $(A \cap B) \cap (C \cap B) = \phi$. Then

$$\Pr[(A \cup C) \mid B] = \frac{\Pr[(A \cup C) \cap B]}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)} + \frac{\Pr(C \cap B)}{\Pr(B)}$$

$$= \Pr(A \mid B) + \Pr(C \mid B)$$

Example of Conditional Probability

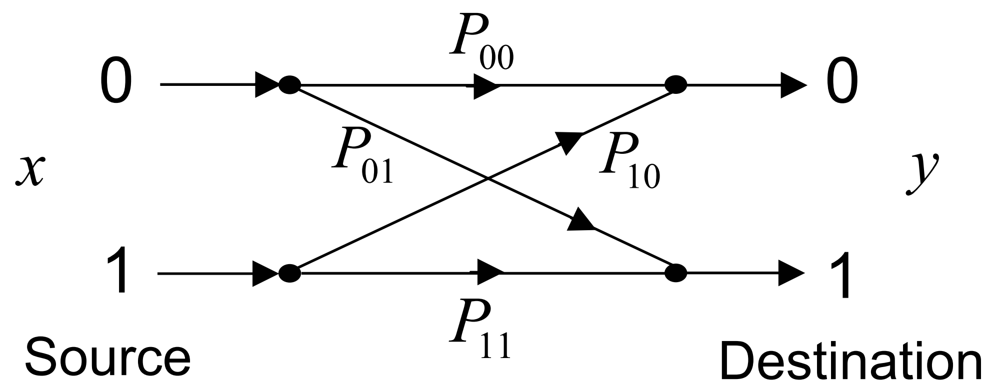
	Bin Number						
Ohms	1	2	3	4	5	6	Total
10	500	0	200	800	1200	1000	3700
100	300	400	600	200	800	0	2300
1000	200	600	200	600	0	1000	2600
Total	1000	1000	1000	1600	2000	2000	8600

What is the probability of drawing a 10 ohm resistor from any bin?
What is the probability of drawing a 100 ohm resistor from bin 3?
A 1000 ohm resistor was drawn. What is the probability that it came from bin 1?

Consider the experiment as one that produces (ohm, bin) as outcome.

Probability & Information Transmission

A binary channel



At the source,

$\Pr(x=0)=p$ and

$\Pr(x=1)=1-p$,

$\Pr(x=0)$ and $\Pr(x=1)$

are called **a priori** probabilities.

The channel, being non-ideal (causing confusions), is characterized by the four conditional probabilities: P_{00}, P_{01}, P_{10}, and P_{11}

$P_{ij}=\Pr(y=j\,|\,x=i)=$ (probability of j received at the destination, i.e., $y=j$, when i was actually sent by the source, i.e., $x=i$)

A posteriori probability –
The probability that i was sent at the source, given that j is received at the destination.

$$\Pr(x=i\,|\,y=j)=\frac{\Pr(y=j\,|\,x=i)\Pr(x=i)}{\Pr(y=j)}$$

Bayes Formula

Is used to find a posteriori probability

$$\Pr(x=i \mid y=j) = \frac{\Pr(y=j \mid x=i)\Pr(x=i)}{\Pr(y=j)}$$

More precisely,

$$\Pr(x=i \mid y=j) = \Pr(\text{symbol } i \text{ was sent, given that } j \text{ is received})$$

$$= \frac{\Pr(y=j \mid x=i)\Pr(x=i)}{\sum_{all\ i}\Pr(y=j \mid x=i)\Pr(x=i)}$$

$$= \frac{P_{ij}\Pr(x=i)}{\sum_{all\ i}P_{ij}\Pr(x=i)} = \frac{P_{ij}\Pr(x=i)}{\Pr(y=j)} =$$ the source portion of contributions that led to the reception of *j*

Applications in Information Transmission

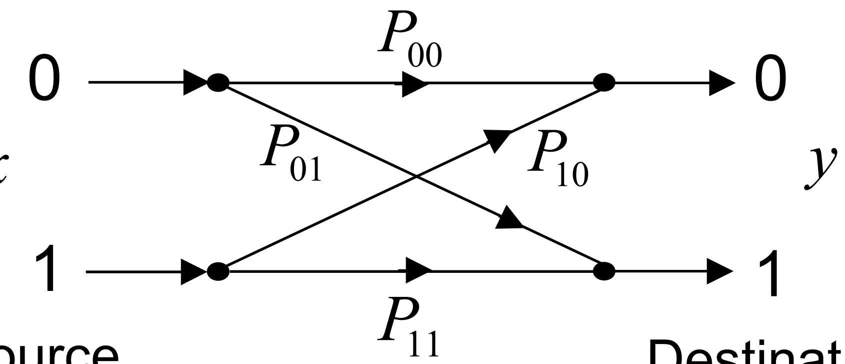
- When symbol 0 is sent, it has a probability of P_{01} being received as symbol 1 – i.e., error occurs.
- When symbol 1 is sent, it has a probability of P_{10} being received as symbol 0 – i.e., error occurs.
- What is the probability of error, regardless of the transmitted symbol?

$$pP_{01} + (1 - p)P_{10}$$

- If we want to make sure of the delivery of information by repeating the same symbol n times, and our decision policy is to go with the majority of the n received symbols, what would be the probability of error in information reception?

Applications in Information Transmission

Repetitive transmission – For every source symbol, we send it three times, i.e., n = 3



x=0

$$P_{00} = P_{11} = 0.9$$
$$P_{01} = P_{10} = 0.1$$

000	$P_{00}P_{00}P_{00}$	y =0	0.729
001	$P_{00}P_{00}P_{01}$	0	0.081
010	$P_{00}P_{01}P_{00}$	0	0.081
100	$P_{01}P_{00}P_{00}$	0	0.081
011	$P_{00}P_{01}P_{01}$	1	0.009
101	$P_{01}P_{00}P_{01}$	1	0.009
110	$P_{01}P_{01}P_{00}$	1	0.009
111	$P_{01}P_{01}P_{01}$	1	0.001

+ } = 0.972 Probability of correct reception

+ } = 0.028 Probability of incorrect reception

Same situation with x=1

The error probability is 0.028, compared to 0.1 with single transmission.

Efficiency of repetitive code is low, though. Other error correction code can have much higher performance.

Independence

- Recall combined experiments and combined trials

  experiment 1 $\Rightarrow$ $\{\text{event } A_{1\bullet}\}$

  experiment 2 $\Rightarrow$ $\{\text{event } A_{2\bullet}\}$

  experiment (1×2) $\Rightarrow$ $\{\text{event } A_{1\bullet}\}\times\{\text{event } A_{2\bullet}\}$

- Two events (of two trials), A_1 and A_2, are **independent** if and only if

  $$\Pr(A_1 \cap A_2) = \Pr(A_1)\Pr(A_2)$$

  $\cap$ means "**and**"

- Extension to n (n > 2) events:

  $$\Pr(A_i \cap A_j \cap \cdots \cap A_k) = \Pr(A_i)\Pr(A_j)\cdots\Pr(A_k)$$

  for any set of integers $\leq n$. Total number of equations to establish independence is $2^n - (n+1)$.

- Be careful not to get confused when A_1 and A_2 are events of the same experiment.

Independence

- If A and B are independent events of the same experiment, what can we say about them?

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)\Pr(B)}{\Pr(B)} = \Pr(A)$$

$$\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{\Pr(A)\Pr(B)}{\Pr(A)} = \Pr(B)$$

That is, the probability assignment of event A has nothing to do with that of event B and *vice versa*.