ECE 3075A
Random Signals

Lecture 4

Random Variables, Probability Distribution Functions

School of Electrical and Computer Engineering
Georgia Institute of Technology
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Random Variables

- A random variable is a numeric function of the outcome of a random experiment.

$$S = \{A\}, \qquad X : A \Rightarrow X(A)$$

Or, equivalently, we consider a probability space defined over the real line. For example, in 2-coin tossing,

$$S = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$$

Define $X(\text{HH}) = 0 \quad X(\text{HT}) = 1 \quad X(\text{TH}) = 3 \quad X(\text{TT}) = 6$

Then $S_X = \{0, 1, 3, 6\}$

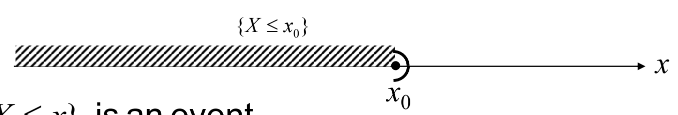
And $\Pr(0) = \Pr(\text{HH}) = 0.25, \qquad \Pr(1) = \Pr(\text{HT}) = 0.25$

$\Pr(3) = \Pr(\text{TH}) = 0.25, \qquad \Pr(6) = \Pr(\text{TT}) = 0.25$

Random Variables (cont'd)

- Types of random variables
  - Discrete random variables: X as a numeric function assumes (equivalently the observation space S_X has) finite set of values.
  - Continuous random variables: X maps the (infinite number of) events to a continuous range of values.
- Purpose of random variables?
  - Many signals are numeric functions of time; uncertainty in the value of the signal function can be addressed with the theory of probability.
  - To bridge between calculus and set operations (σ algebra); being able to do so broadens our ability to handle the assignment of probability to (infinite) event.

Events Associated with R.V.s



$\{X \le x\}$ is an event.

- X is a random variable, the value of which is to be observed through trials;
- x is an arbitrary real number;
- The event $\{X \le x\}$ means that the observed value of the random variable X is less than or equal to the specified value x;
- A real value x thus defines an event;
- The **probability** of this event is $\Pr(X \le x)$ which is a **function of** x.

Probability Distribution Function

- $\Pr(X \le x)$ is a **function** of x.
  $F_X(x) = \Pr(X \le x)$ is the **probability distribution function** defined over all x.

$\{X \le -\infty\} = \phi, \quad F_X(-\infty) = 0$

$\{X \le \infty\}$ is always true, a sure event, thus, $F_X(\infty) = 1$

$\{X \le -\infty\} \subset \{X \le x\} \subset \{X \le \infty\}$ for $-\infty < x < \infty$

$\Rightarrow \quad 0 \le F_X(x) \le 1$

» If $x_1 < x_2, \quad \{X \le x_1\} \subset \{X \le x_2\}, \quad \{X \le x_1\} \cap \{X \le x_2\} = \{X \le x_1\}$

$\{X \le x_2\} = \{X \le x_1\} \cup \{x_1 < X \le x_2\}$, and $\{X \le x_1\} \cap \{x_1 < X \le x_2\} = \phi$

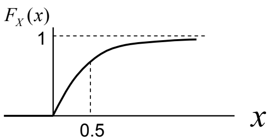
$\therefore \quad \Pr(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

$F_X(x)$ is a non-decreasing function of x.

Example

A random variable has a probability distribution function given by

$$F_X(x) = 0 \qquad -\infty < x \le 0$$
$$= 1 - e^{-2x} \qquad 0 \le x < \infty$$



Find

a) The probability that $X > 0.5$

$$\Pr(X > 0.5) = 1 - F_X(0.5) = 1 - (1 - e^{-1}) = e^{-1} = 0.3679$$

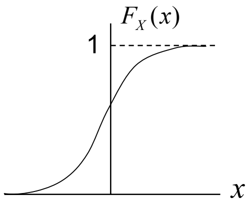
b) The probability that $X \le 0.25$

$$\Pr(X \le 0.25) = F_X(0.25) = 1 - e^{-0.5} = 1 - 0.6065 = 0.3935$$

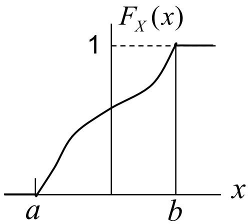
c) The probability that $0.3 < X \le 0.7$

$$\Pr(0.3 < X \le 0.7) = F_X(0.7) - F_X(0.3)$$
$$= 1 - e^{-1.4} - (1 - e^{-0.6}) = 0.5488 - 0.2466 = 0.3022$$

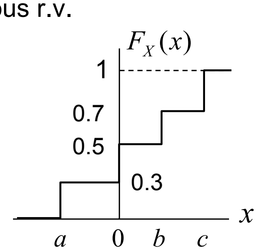
Example of Probability Distribution Function



Continuous r.v.



Continuous r.v. within limits a and b



Discrete r.v. assuming values a, 0, b and c

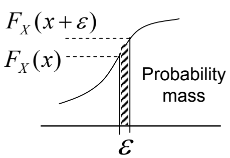
Probability Density Functions

- The slope of the probability distribution function at x represents the incremental probability at that point and thus gives the sense of how likely $X = x$ might be.

$$f_X(x) = \lim_{\varepsilon \to 0} \frac{F_X(x+\varepsilon) - F_X(x+\varepsilon)}{\varepsilon} = \frac{dF_X(x)}{dx}$$

$$f_X(x)dx = \Pr(x < X \le x + dx)$$

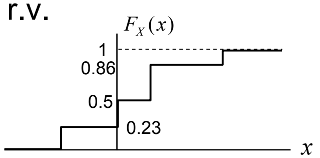
is the probability mass at x.

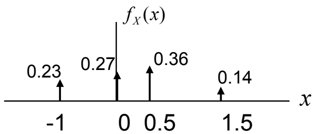


- The derivative is called the probability density function (pdf). Pdf is **non-negative**. In the case of discrete distributions, the pdf consists of delta functions at those realizable values, each having an area equal to the corresponding magnitude of probability.

Examples of PDFs

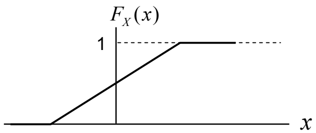
- Distribution of a discrete r.v.

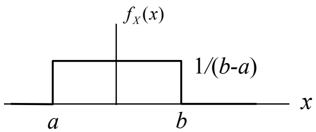




$$f_X(x)=\sum_i p_i\delta(x-x_i)$$

- Continuous r.v. – **uniform distribution**





$$f_X(x)=(b-a)^{-1},\quad a\le x\le b$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = F_X(\infty)=1$$

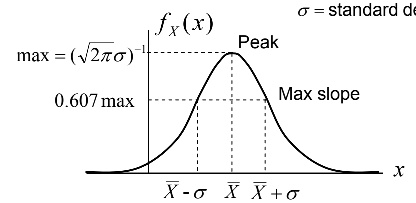
Gaussian Random Variable

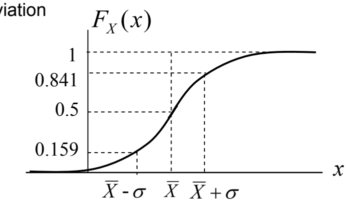
- A r.v. X is gaussian if its pdf is of the form

$$f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left[\frac{-(x-\overline{X})^2}{2\sigma^2}\right],\quad -\infty<x<\infty$$

where $\overline{X}$ and σ^2 are called the mean and variance, respectively.

σ = standard deviation





✓ Also called normal distribution, denoted as $\mathcal{N}(x;\overline{X},\sigma^2)$

✓ Pdf has a single peak.

✓ $\delta(x-\overline{X})=\lim_{\sigma\to 0}(\sqrt{2\pi}\sigma)^{-1}\exp\left[-(x-\overline{X})^2/(2\sigma^2)\right]$, a good representation for a delta function because a Gaussian pdf is infinitely differentiable.

Gaussian Distribution

- Often expressed in zero mean and unity variance form with

$$u=\frac{x-\overline{X}}{\sigma},\qquad f_U(u)=\frac{1}{\sqrt{2\pi}}\exp\left[\frac{-u^2}{2}\right],\quad -\infty<u<\infty$$

Or equivalently, define $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left[\frac{-u^2}{2}\right]du,$

Then, $F_X(x)=\Phi\left(\frac{x-\overline{X}}{\sigma}\right)$

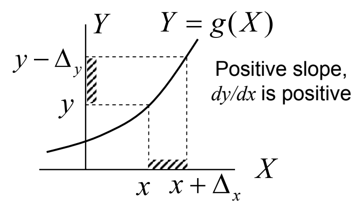
- The Q-function $Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^{\infty}\exp\left[\frac{-u^2}{2}\right]du=1-\Phi(x)$
- The error function $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp\left[-u^2\right]du$

$$Q(x)=\frac{1}{2}\left[1-erf\left(\frac{x}{\sqrt{2}}\right)\right]\approx\left[\frac{1}{(1-a)x+a\sqrt{x^2+b}}\right]\frac{\exp(-x^2/2)}{\sqrt{2\pi}},\quad x\ge 0$$

$$a=0.339,\quad b=5.510$$

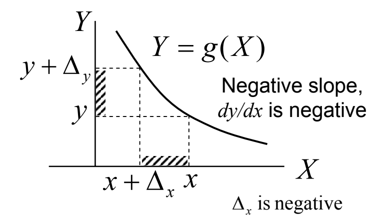
Functions of Random Variable

- X is a random variable with pdf $f_X(x)$.
- Y is a monotonic function of X; $Y=g(X)$. Find $f_Y(y)$.



$\Pr(x<X\le x+\Delta_x)=\Pr(y<Y\le y+\Delta_y)$

$f_X(x)dx=f_Y(y)dy$ or $f_Y(y)=f_X(x)\frac{dx}{dy}$



Δ_x is negative

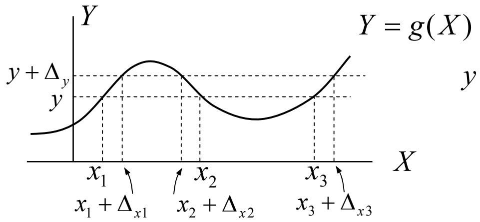
$\Pr(x+\Delta_x<X\le x)=\Pr(y<Y\le y+\Delta_y)$

$f_X(x)dx=f_Y(y)(-dy)$ or $f_Y(y)=f_X(x)\frac{-dx}{dy}$

Hence, $f_Y(y)=f_X(x)\left|\frac{dx}{dy}\right|$

Expressed in y with $g^{-1}(y)=x$, $f_Y(y)=f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$

Non-Monotonic Functions of R.V.



$$y = g(x_1) = g(x_2) = g(x_3)$$

$$\Pr(y < Y \le y + \Delta_y)$$

$$= \Pr(x_1 < X \le x_1 + \Delta_{x1})$$

$$+ \Pr(x_2 + \Delta_{x2} < X \le x_2)$$

$$+ \Pr(x_3 < X \le x_3 + \Delta_{x3})$$

$$f_Y(y) = \sum_{\text{for all } x = g^{-1}(y)} f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

Example:

$$Y = X^2 \text{ or } X = \sqrt{Y}, \quad \left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{y}}, \quad \text{but for any } y > 0, \; x = \pm\sqrt{y}$$

Therefore,

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right], \quad y \ge 0; \qquad = 0, \quad y < 0$$