

ECE 3075A
Random Signals

Lecture 4

**Random Variables, Probability Distribution
Functions**

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Random Variables

- A random variable is a numeric function of the outcome of a random experiment.

$$S = \{A\}, \quad X : A \Rightarrow X(A)$$

Or, equivalently, we consider a probability space defined over the real line. For example, in 2-coin tossing,

$$S = \{HH, HT, TH, TT\}$$

Define $X(HH) = 0$ $X(HT) = 1$ $X(TH) = 3$ $X(TT) = 6$

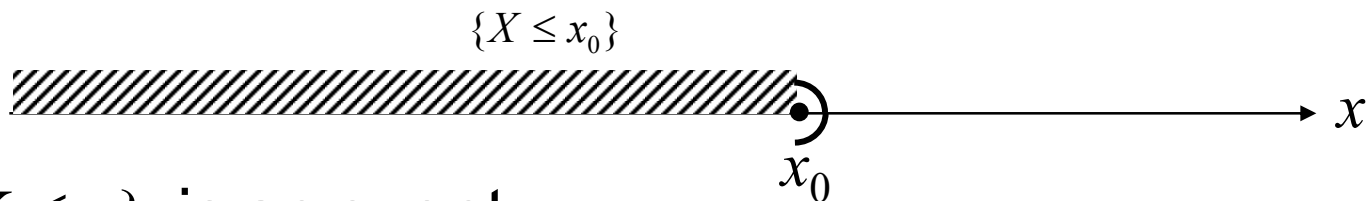
Then $S_X = \{0, 1, 3, 6\}$

And $\Pr(0) = \Pr(HH) = 0.25,$ $\Pr(1) = \Pr(HT) = 0.25$
 $\Pr(3) = \Pr(TH) = 0.25,$ $\Pr(6) = \Pr(TT) = 0.25$

Random Variables (cont'd)

- Types of random variables
 - Discrete random variables: X as a numeric function assumes (equivalently the observation space \mathcal{S}_X has) finite set of values.
 - Continuous random variables: X maps the (infinite number of) events to a continuous range of values.
- Purpose of random variables?
 - Many signals are numeric functions of time; uncertainty in the value of the signal function can be addressed with the theory of probability.
 - To bridge between calculus and set operations (σ algebra); being able to do so broadens our ability to handle the assignment of probability to (infinite) event.

Events Associated with R.V.s



$\{X \leq x\}$ is an event.

- X is a random variable, the value of which is to be observed through trials;
- x is an arbitrary real number;
- The event $\{X \leq x\}$ means that the observed value of the random variable X is less than or equal to the specified value x ;
- A real value x thus defines an event;
- The **probability** of this event is $\Pr(X \leq x)$ which is a **function of x** .

Probability Distribution Function

- $\Pr(X \leq x)$ is a **function** of x .

$F_X(x) = \Pr(X \leq x)$ is the **probability distribution function** defined over all x .

$$\{X \leq -\infty\} = \phi, \quad F_X(-\infty) = 0$$

$\{X \leq \infty\}$ is always true, a sure event, thus, $F_X(\infty) = 1$

$$\{X \leq -\infty\} \subset \{X \leq x\} \subset \{X \leq \infty\} \quad \text{for } -\infty < x < \infty$$

$$\Rightarrow 0 \leq F_X(x) \leq 1$$

» If $x_1 < x_2$, $\{X \leq x_1\} \subset \{X \leq x_2\}$, $\{X \leq x_1\} \cap \{X \leq x_2\} = \{X \leq x_1\}$

$$\{X \leq x_2\} = \{X \leq x_1\} \cup \{x_1 < X \leq x_2\}, \quad \text{and } \{X \leq x_1\} \cap \{x_1 < X \leq x_2\} = \phi$$

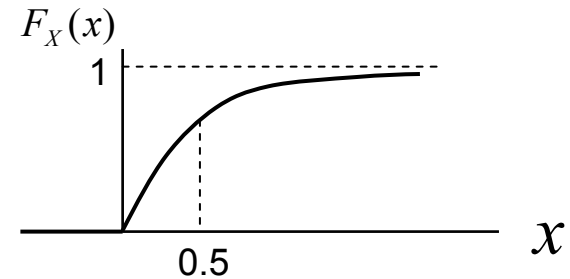
$$\therefore \Pr(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$$

$F_X(x)$ is a non-decreasing function of x .

Example

A random variable has a probability distribution function given by

$$F_X(x) = 0 \quad -\infty < x \leq 0$$
$$= 1 - e^{-2x} \quad 0 \leq x < \infty$$



Find

a) The probability that $X > 0.5$

$$\Pr(X > 0.5) = 1 - F_X(0.5) = 1 - (1 - e^{-1}) = e^{-1} = 0.3679$$

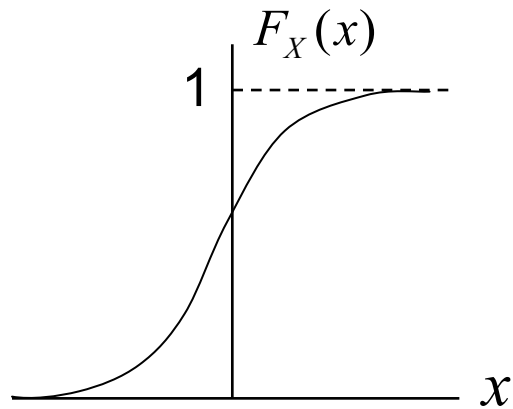
b) The probability that $X \leq 0.25$

$$\Pr(X \leq 0.25) = F_X(0.25) = 1 - e^{-0.5} = 1 - 0.6065 = 0.3935$$

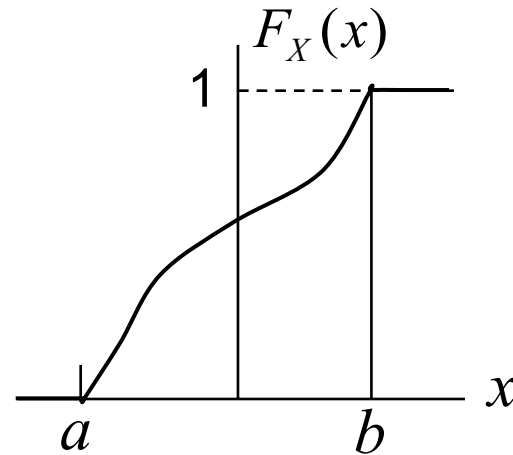
c) The probability that $0.3 < X \leq 0.7$

$$\Pr(0.3 < X \leq 0.7) = F_X(0.7) - F_X(0.3)$$
$$= 1 - e^{-1.4} - (1 - e^{-0.6}) = 0.5488 - 0.2466 = 0.3022$$

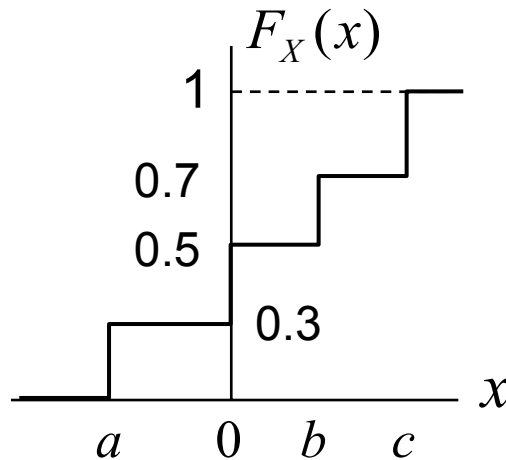
Example of Probability Distribution Function



Continuous r.v.



Continuous r.v. within limits a and b



Discrete r.v. assuming values $a, 0, b$ and c

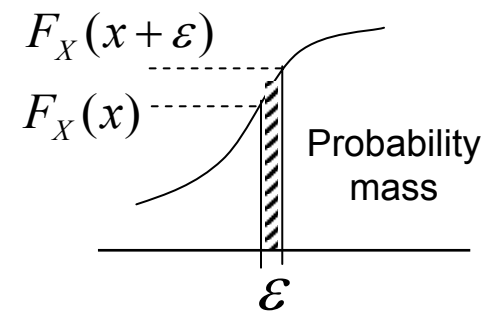
Probability Density Functions

- The slope of the probability distribution function at x represents the incremental probability at that point and thus gives the sense of how likely $X = x$ might be.

$$f_X(x) = \lim_{\varepsilon \rightarrow 0} \frac{F_X(x + \varepsilon) - F_X(x)}{\varepsilon} = \frac{dF_X(x)}{dx}$$

$$f_X(x)dx = \Pr(x < X \leq x + dx)$$

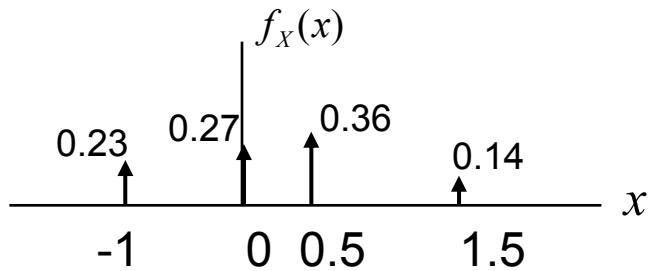
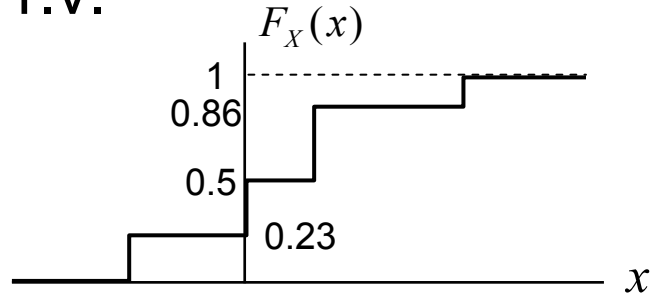
is the probability mass at x .



- The derivative is called the probability density function (pdf). Pdf is **non-negative**. In the case of discrete distributions, the pdf consists of delta functions at those realizable values, each having an area equal to the corresponding magnitude of probability.

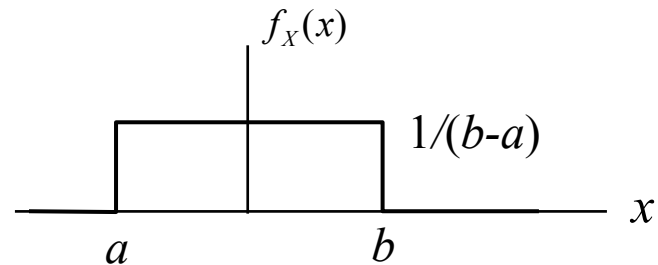
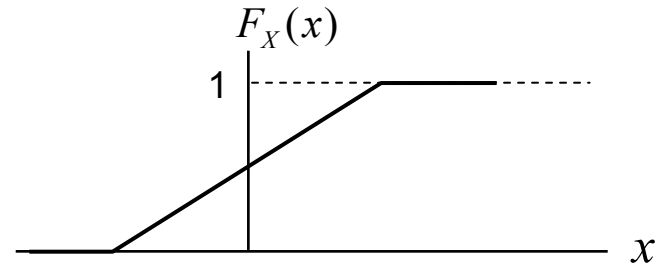
Examples of PDFs

- Distribution of a discrete r.v.



$$f_X(x) = \sum_i p_i \delta(x - x_i)$$

- Continuous r.v. – **uniform distribution**



$$f_X(x) = (b - a)^{-1}, \quad a \leq x \leq b$$

$$\int_{-\infty}^{\infty} f_X(x) dx = F_X(\infty) = 1$$

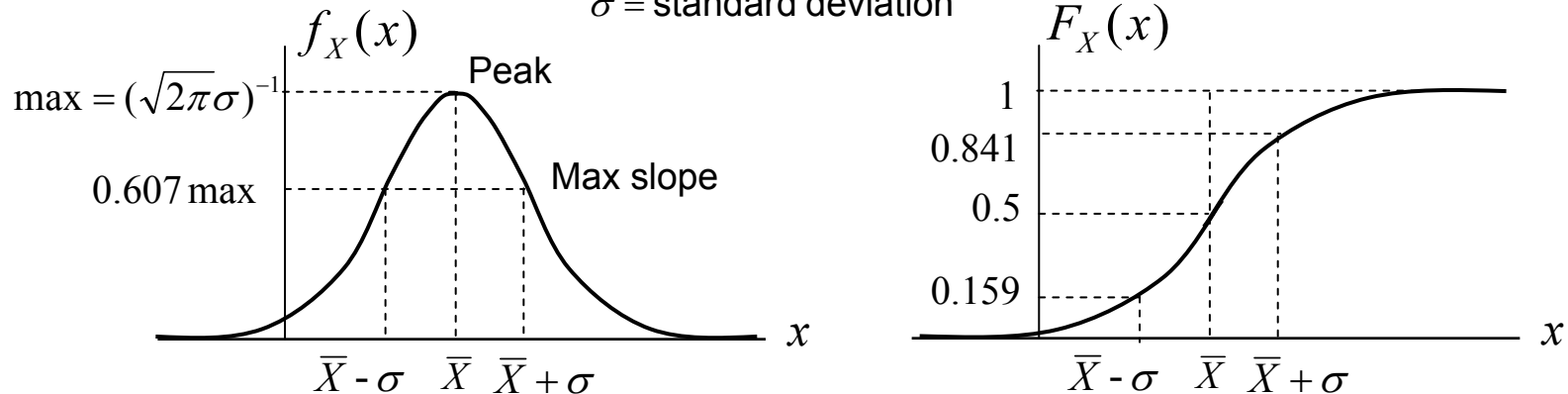
Gaussian Random Variable

- A r.v. X is gaussian if its pdf is of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x - \bar{X})^2}{2\sigma^2}\right], \quad -\infty < x < \infty$$

where \bar{X} and σ^2 are called the mean and variance, respectively.

σ = standard deviation



- ✓ Also called normal distribution, denoted as $\mathcal{N}(x; \bar{X}, \sigma^2)$
- ✓ Pdf has a single peak.
- ✓ $\delta(x - \bar{X}) = \lim_{\sigma \rightarrow 0} (\sqrt{2\pi}\sigma)^{-1} \exp[-(x - \bar{X})^2 / (2\sigma^2)]$, a good representation for a delta function because a Gaussian pdf is infinitely differentiable.

Gaussian Distribution

- Often expressed in zero mean and unity variance form with

$$u = \frac{x - \bar{X}}{\sigma}, \quad f_U(u) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2}\right], \quad -\infty < u < \infty$$

Or equivalently, define $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left[-\frac{u^2}{2}\right] du,$

Then, $F_X(x) = \Phi\left(\frac{x - \bar{X}}{\sigma}\right)$

- The Q -function $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left[-\frac{u^2}{2}\right] du = 1 - \Phi(x)$

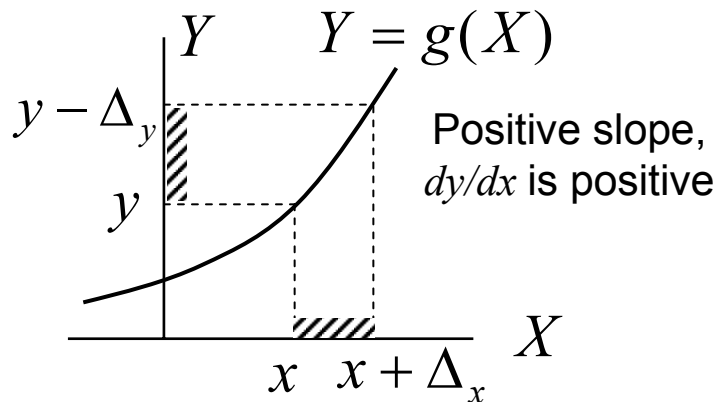
- The error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp[-u^2] du$

$$Q(x) = \frac{1}{2} \left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \right] \approx \left[\frac{1}{(1-a)x + a\sqrt{x^2 + b}} \right] \frac{\exp(-x^2/2)}{\sqrt{2\pi}}, \quad x \geq 0$$

$$a = 0.339, \quad b = 5.510$$

Functions of Random Variable

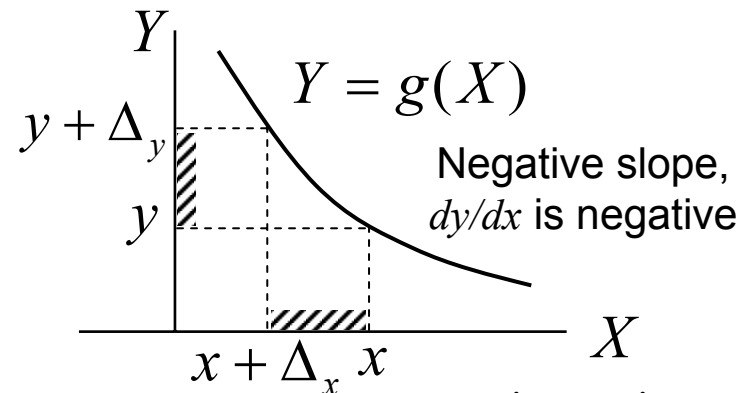
- X is a random variable with pdf $f_X(x)$.
- Y is a monotonic function of X ; $Y = g(X)$. Find $f_Y(y)$.



$$\Pr(x < X \leq x + \Delta_x) = \Pr(y < Y \leq y + \Delta_y)$$

$$f_X(x)dx = f_Y(y)dy \quad \text{or} \quad f_Y(y) = f_X(x) \frac{dx}{dy}$$

$$\text{Hence, } f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$



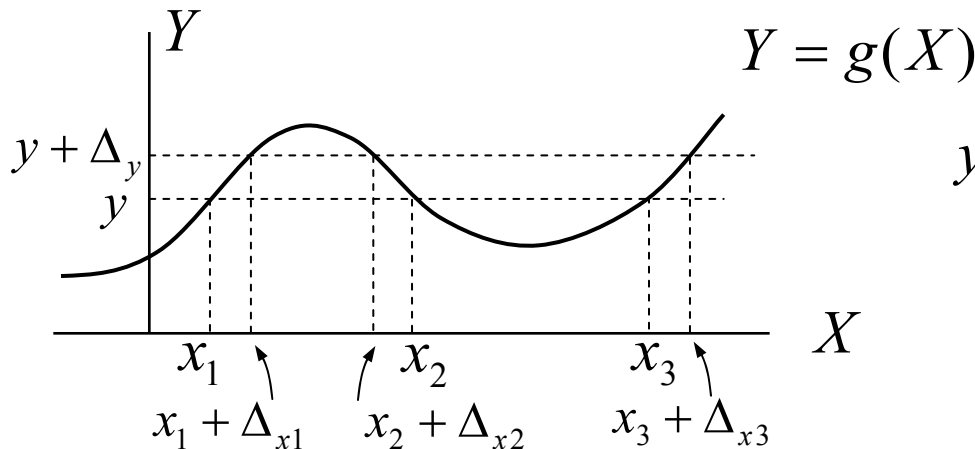
Δ_x is negative

$$\Pr(x + \Delta_x < X \leq x) = \Pr(y < Y \leq y + \Delta_y)$$

$$f_X(x)dx = f_Y(y)(-dy) \quad \text{or} \quad f_Y(y) = f_X(x) \frac{-dx}{dy}$$

$$\text{Expressed in } y \text{ with } g^{-1}(y) = x, \quad f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

Non-Monotonic Functions of R.V.



$$y = g(x_1) = g(x_2) = g(x_3)$$

$$\Pr(y < Y \leq y + \Delta_y)$$

$$= \Pr(x_1 < X \leq x_1 + \Delta_{x1})$$

$$+ \Pr(x_2 + \Delta_{x2} < X \leq x_2)$$

$$+ \Pr(x_3 < X \leq x_3 + \Delta_{x3})$$

$$f_Y(y) = \sum_{\text{for all } x=g^{-1}(y)} f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

Example:

$$Y = X^2 \text{ or } X = \sqrt{Y}, \quad \left| \frac{dx}{dy} \right| = \frac{1}{2\sqrt{y}}, \quad \text{but for any } y > 0, \quad x = \pm\sqrt{y}$$

Therefore,

$$f_Y(y) = \frac{1}{2\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})], \quad y \geq 0; \quad = 0, \quad y < 0$$