ECE 3075A
Random Signals

Lecture 5
Probability Distribution Functions, Expectations, and Moments

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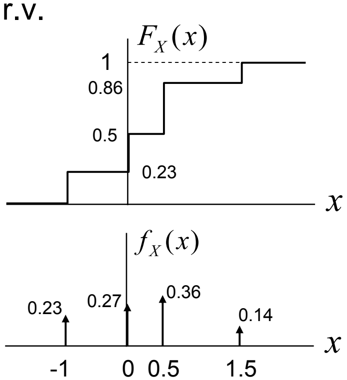
Common Probability Distribution Functions

- Uniform distribution
- Gaussian or normal distribution
- Binomial distribution
- Poisson distribution
- Exponential and double exponential distribution
- Cauchy distribution
- Log normal distribution
- Mixed distribution and mixture density functions

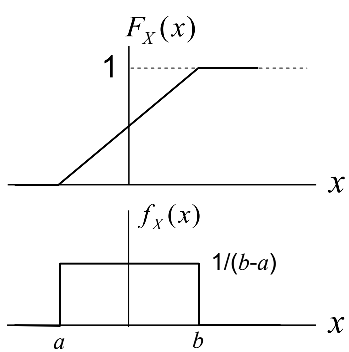
Examples of PDFs

- Distribution of a discrete r.v.



$$f_X(x) = \sum_i p_i \delta(x - x_i)$$

- Continuous r.v. – uniform distribution



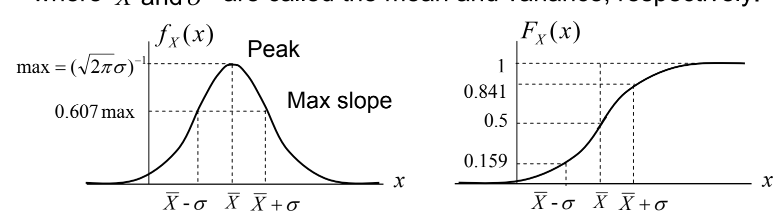
$$f_X(x) = (b-a)^{-1}, \quad a \le x \le b$$

Gaussian Random Variable

- A r.v. X is gaussian if its pdf is of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\bar{X})^2}{2\sigma^2}\right], \quad -\infty < x < \infty$$

where $\bar{X}$ and σ^2 are called the mean and variance, respectively.



- ✓ Also called normal distribution, denoted as $\mathcal{N}(x; \bar{X}, \sigma^2)$
- ✓ Pdf has single peak.
- ✓ $\delta(x - \bar{X}) = \lim_{\sigma \to 0} (\sqrt{2\pi}\sigma)^{-1} \exp\left[-(x-\bar{X})^2/(2\sigma^2)\right]$, a good representation for a delta function because a gaussian pdf is infinitely differentiable.

Binomial Distribution

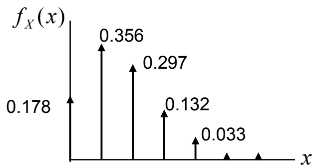
- A discrete distribution

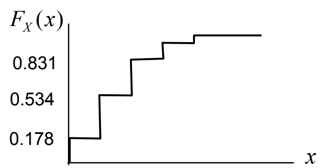
  Binomial density function $\binom{n}{k} = \frac{n!}{k!(n-k)!} =$ binomial coeff.

$$f_X(x) = \sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(x-k) \quad \text{for } 0 < p < 1 \text{ and integer } n.$$

$n = 6,$

$p = 0.25$





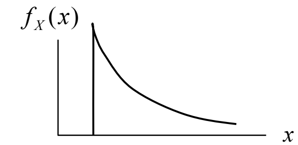
Recall that binomial density function approaches gaussian with mean np and variance $np(1-p)$ when n is large and n, p satisfy certain conditions.

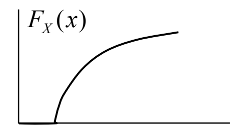
Exponential Density and Distribution

- Exponential pdf and distribution function – for b>0 and -∞<a< ∞,

$$f_X(x) = \begin{cases} b^{-1} \exp[-(x-a)/b], & x > a \\ 0, & x < a \end{cases}$$

$$F_X(x) = \begin{cases} 1 - \exp[-(x-a)/b], & x > a \\ 0, & x < a \end{cases}$$





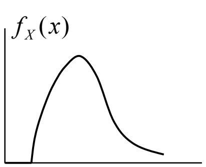
Useful in
- describing raindrop size when a large number of rainstorm measurements are made
- modeling the fluctuation in strength in received radar signal from a certain types of aircraft
- modeling the interval of arrival time of a certain traffic

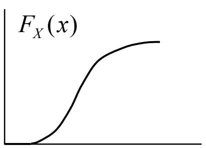
Rayleigh Density and Distribution

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a)\exp\left[\frac{-(x-a)^2}{b}\right], & x \geq a \\ 0, & x < a \end{cases}$$

$$F_X(x) = \begin{cases} 1 - \exp\left[\frac{-(x-a)^2}{b}\right], & x \geq a \\ 0, & x < a \end{cases}$$

$b > 0, \quad -\infty < a < \infty$





- Was derived as the density function of the envelop of the sum of many sine waves of different frequencies
- Also used in modeling coordinate errors $\Delta_r = \sqrt{\Delta_x^2 + \Delta_y^2}$ where Δ_x and Δ_y are independent gaussian r.v. with zero mean and equal variance.

Poisson Distribution

$$f_X(x) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \delta(x-k), \quad \lambda > 0$$

Used in describing:
- Number of defective units in a sample taken from a production line
- Number of telephone calls made during a period of time
- Number of customers coming to a store within a period of time
- Number of electrons emitted from a section of cathode in a given time interval

λ is the average number of “arrivals”, “calls”, or “emissions” within a unit time interval.

Other Density & Distribution Functions

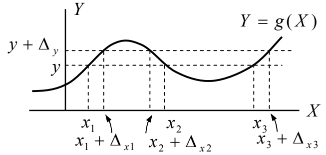
- Cauchy $$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$
$$F_X(x) = \int_{-\infty}^{x} \frac{dx}{\pi(1+x^2)} = \frac{1}{\pi}\left[\arctan x + \frac{\pi}{2}\right], \quad -\infty < x < \infty$$
- Mixed Distribution or Mixture
$$f_X(x) = \sum_{k=1}^{n} c_k p_k(x), \quad \sum_{k=1}^{n} c_k = 1, \quad \text{and} \quad F_X(x) = \sum_{k=1}^{n} c_k P_k(x)$$

Where $p_k(x)$ is some kernel density function such as Gaussian,

then, $f_X(x) = \sum_{k=1}^{n} c_k \mathcal{N}_k(x; \eta_k, \sigma_k^2)$ and $\overline{X} = \sum_{k=1}^{n} c_k \eta_k$

Functions of Random Variable (Reminder)

- X is a random variable with pdf $f_x(x)$.
- Y is a monotonic function of X; $Y = g(X)$. Find $f_y(y)$.
$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$
Expressed in y with $g^{-1}(y) = x$, $\quad f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$
- If Y is non-monotonic function of X,
$$f_Y(y) = \sum_{\text{for all } x = g^{-1}(y)} f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$$



Example: $Y = X^2$ or $X = \sqrt{Y}$, $\left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{y}}$, but for any $y > 0$, $x = \pm\sqrt{y}$

Therefore, $f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right] \quad y \geq 0; \quad = 0, \quad y < 0$

Mean Values and Moments

- Mean or expected value of a random variable
$$\overline{X} = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$
- Mean or expected value of a function of a random variable $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$
- N^{th} moments & central moments of a r.v.
$$\overline{X^n} = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx \qquad \overline{(X-\overline{X})^n} = E[(X-\overline{X})^n] = \int_{-\infty}^{\infty} (x-\overline{X})^n f_X(x) dx$$

When $n = 2$,

$\overline{X^2} = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$ is called the mean-square value

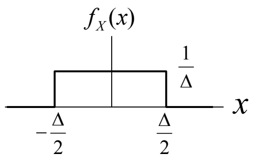
and $\sigma^2 = E[(X-\overline{X})^2] = \int_{-\infty}^{\infty} (x-\overline{X})^2 f_X(x) dx$ is the variance.

$$\sigma^2 = E[(X-\overline{X})^2] = E[X^2 - 2X\overline{X} + \overline{X}^2] = E[X^2] - 2\overline{X}^2 + \overline{X}^2 = E[X^2] - \overline{X}^2$$

Moments of Uniform R.V.

$$f_X(x) = \frac{1}{\Delta}\left[u\left(x+\frac{\Delta}{2}\right) - u\left(x-\frac{\Delta}{2}\right)\right]$$

$$u(n) = \text{"step function"} = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$\overline{X} = \int_{-\Delta/2}^{\Delta/2} x f_X(x)\,dx = \int_{-\Delta/2}^{\Delta/2} \frac{x}{\Delta} dx = \frac{x^2}{2\Delta}\Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta}{8} - \frac{\Delta}{8} = 0$$

$$\overline{X^2} = \int_{-\Delta/2}^{\Delta/2} x^2 f_X(x)\,dx = \int_{-\Delta/2}^{\Delta/2} \frac{x^2}{\Delta} dx = \frac{x^3}{3\Delta}\Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{24} + \frac{\Delta^2}{24} = \frac{\Delta^2}{12}$$

$$\sigma^2 = \overline{X^2} - (\overline{X})^2 = \overline{X^2} = \frac{\Delta^2}{12} \qquad \overline{X^3} = \int_{-\Delta/2}^{\Delta/2} \frac{x^3}{\Delta} dx = \frac{x^4}{4\Delta}\Big|_{-\Delta/2}^{\Delta/2} = 0$$

The n^{th} moment is zero for odd n.

Example – Exercise 2-4.2

A random variable X has a pdf of the form

$$f_X(x) = 0.25[u(x) - u(x-4)]$$

For the r.v. $Y = X^2$, find

a) The mean value

$$\overline{Y} = \int_0^4 x^2 \bullet 0.25\,dx = x^3/12\Big|_{x=0}^4 = 16/3$$

b) The mean-square value

$$\overline{Y^2} = \int_0^4 x^4 \bullet 0.25\,dx = x^5/20\Big|_{x=0}^4 = 256/5$$

c) The variance.

$$\sigma_Y^2 = \overline{(Y-\overline{Y})^2} = E[Y^2] - \overline{Y}^2 = (256/5) - (16/3)^2$$
$$= 256\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{1024}{45}$$

Entropy – Average Information

- Let $\{p_i\}_{i=1}^L$ be the (a priori) probability associated with a source which puts out symbols $\{X = x_i\}_{i=1}^L$;
- Define information in symbol x_i as $-\log_2 p_i$;
- The average information is called entropy and is defined as

$$H = E_X[-\log_2 p_i] = -\sum_{i=1}^{L} p_i \log_2 p_i$$

For example, if $L=2$, and $x_1 = 0$, and $x_2 = 1$, then it is a binary source with symbols 0 and 1. And if $p_1 = p_2 = 0.5$, then every symbol the source puts out carries one **bit** of information.
if $p_1 \neq p_2$, say, $p_1 = 0.8$ and $p_2 = 0.2$, then the average information in each symbol is less than one bit.

Conditional Probability Distribution

We define the conditional probability the same as before.

$$F_X(x|M) = \Pr(X \le x | M) = \frac{\Pr(X \le x, M)}{\Pr(M)}, \quad \Pr(M) > 0$$

If we use the event mapping concept, $\Pr(X \le x, M)$ is the probability of all the outcomes which realize both events $X(\xi) \le x$ and $\xi \in M$.

If $M = \{X \le m\}$, $F_X(x|M) = \Pr(X \le x | X \le m) = \dfrac{\Pr(X \le x, X \le m)}{\Pr(X \le m)}$

If $x \le m$, $F_X(x|M) = \dfrac{\Pr(X \le x)}{\Pr(X \le m)} = \dfrac{F_X(x)}{F_X(m)}$ If $x \ge m$, $F_X(x|M) = \dfrac{\Pr(X \le m)}{\Pr(X \le m)} = 1$

Conditional probability density function has all the properties of a usual pdf. $f_X(x|M) = \dfrac{dF_X(x|M)}{dx}$

Conditional Expected Value

- The expected value of X, given event M, is

$$E[X|M] = \int_{-\infty}^{\infty} x f_X(x|M)\,dx$$

if $M = \{X \le m\}$, $f_X(x|X \le m) = \begin{cases} \dfrac{f_X(x)}{F_X(m)}, & x < m \\ 0, & x \ge m \end{cases}.$

Thus, $E[X|X \le m] = \dfrac{\int_{-\infty}^{m} x f_X(x)\,dx}{\int_{-\infty}^{m} f_X(x)\,dx}$

which is the expected value of X when X is constrained to the set/event $\{X \le m\}$.

Useful Inequalities

- Chebychev Inequality

$$\Pr\{|X-\overline{X}| \geq \varepsilon\} \leq \sigma_X^2 / \varepsilon^2$$

$$\Pr\{|X-\overline{X}| \geq \varepsilon\} = \int_{-\infty}^{\overline{X}-\varepsilon} f_X(x)\,dx + \int_{\overline{X}-\varepsilon}^{\infty} f_X(x)\,dx = \int_{x\in B} f_X(x)dx$$

where $B = \left\{R - \{|x-\overline{X}| < \varepsilon\}\right\}$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-\overline{X})^2 f_X(x)\,dx \geq \int_{x\in B} (x-\overline{X})^2 f_X(x)\,dx \geq \varepsilon^2 \int_{x\in B} f_X(x)\,dx$$

$$= \varepsilon^2 \Pr\{|X-\overline{X}| \geq \varepsilon\}$$

- Markov Inequality: for non-negative r.v. X,

$$\Pr\{X \geq a\} \leq E[X]/a, \quad a > 0$$

