

**ECE 3075A**  
***Random Signals***

**Lecture 5**

**Probability Distribution Functions, Expectations, and  
Moments**

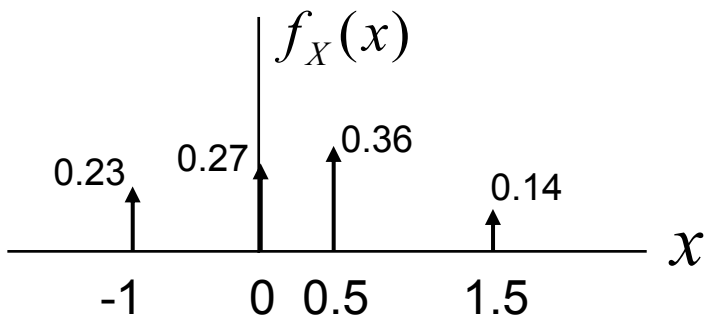
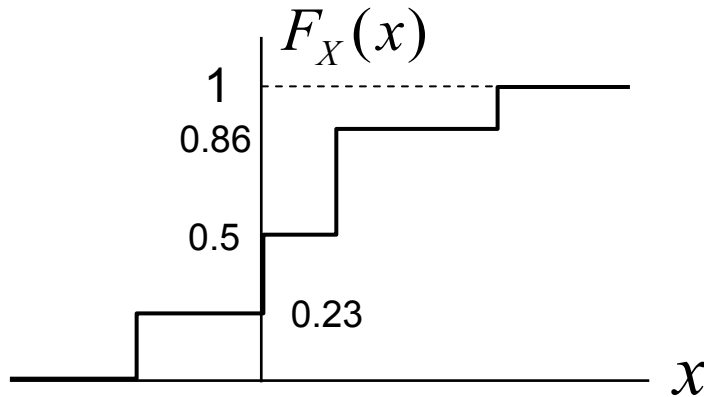
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# Common Probability Distribution Functions

- Uniform distribution
- Gaussian or normal distribution
- Binomial distribution
- Poisson distribution
- Exponential and double exponential distribution
- Cauchy distribution
- Log normal distribution
- Mixed distribution and mixture density functions

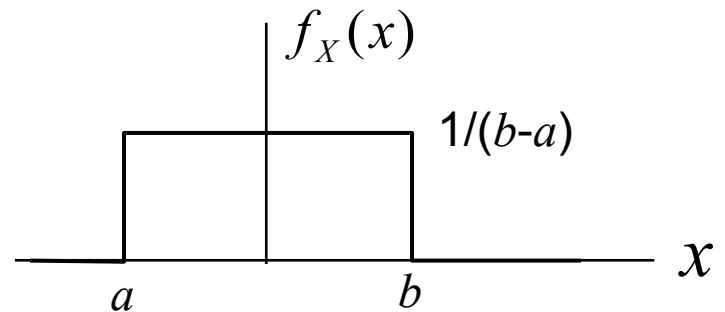
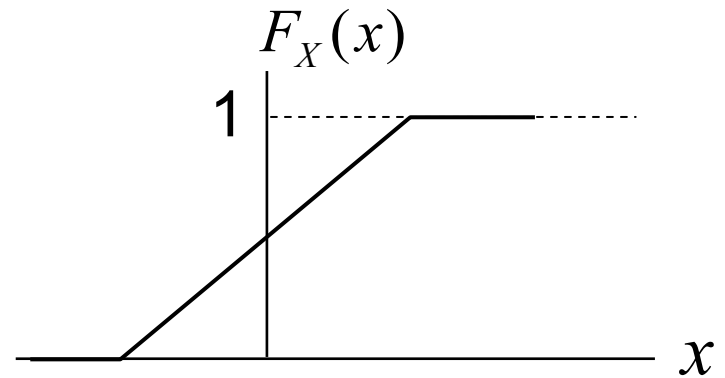
# Examples of PDFs

- Distribution of a discrete r.v.



$$f_X(x) = \sum_i p_i \delta(x - x_i)$$

- Continuous r.v. – uniform distribution



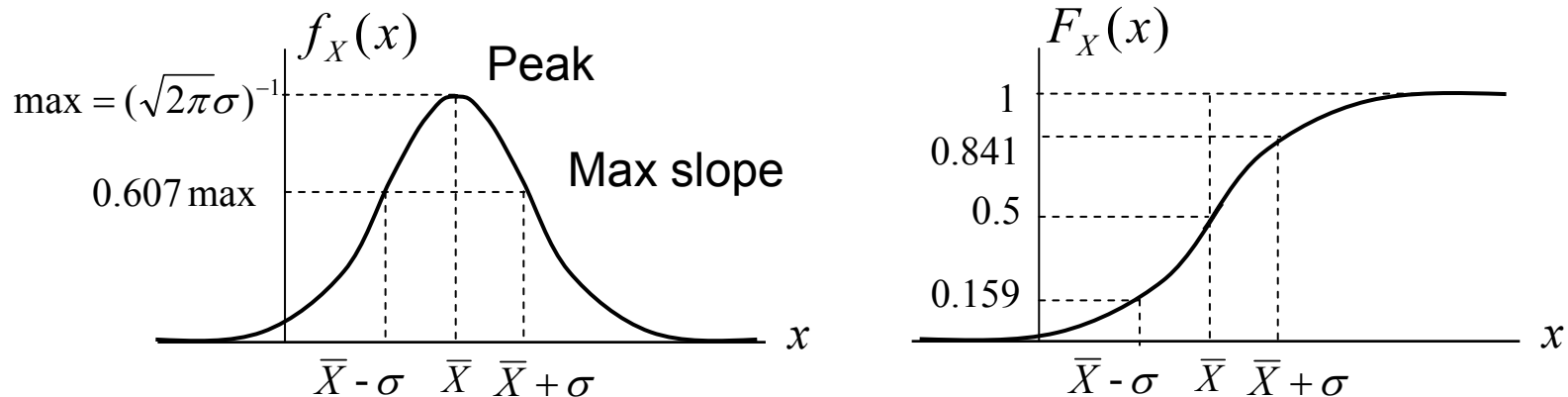
$$f_X(x) = (b - a)^{-1}, \quad a \leq x \leq b$$

# Gaussian Random Variable

- A r.v.  $X$  is gaussian if its pdf is of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x - \bar{X})^2}{2\sigma^2}\right], \quad -\infty < x < \infty$$

where  $\bar{X}$  and  $\sigma^2$  are called the mean and variance, respectively.



- ✓ Also called normal distribution, denoted as  $\mathcal{N}(x; \bar{X}, \sigma^2)$
- ✓ Pdf has single peak.
- ✓  $\delta(x - \bar{X}) = \lim_{\sigma \rightarrow 0} (\sqrt{2\pi}\sigma)^{-1} \exp[-(x - \bar{X})^2 / (2\sigma^2)]$ , a good representation for a delta function because a gaussian pdf is infinitely differentiable.

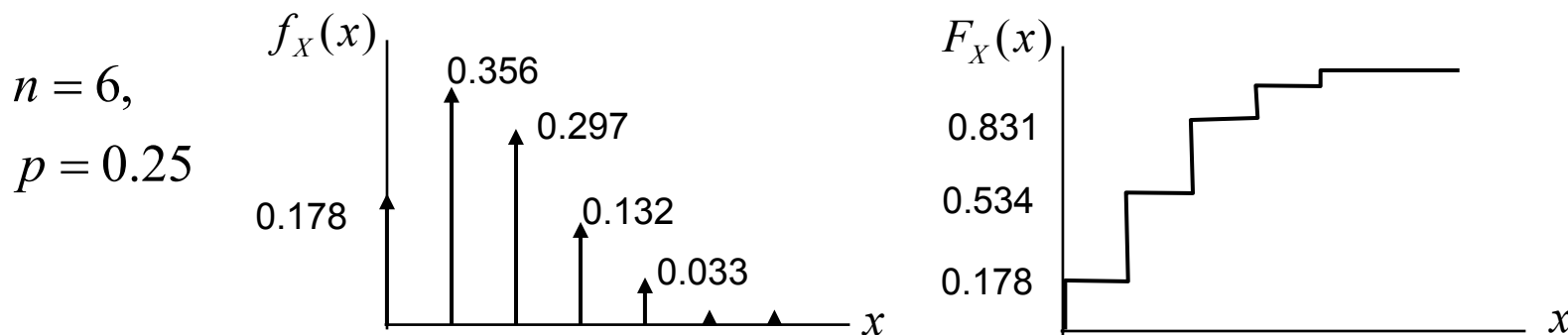
# Binomial Distribution

- A discrete distribution

Binomial density function

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \text{binomial coeff.}$$

$$f_X(x) = \sum_{k=1}^n \binom{n}{k} p^k (1-p)^{n-k} \delta(x-k) \quad \text{for } 0 < p < 1 \text{ and integer } n.$$



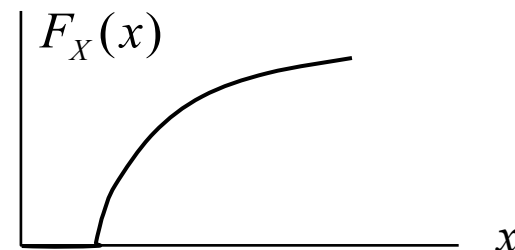
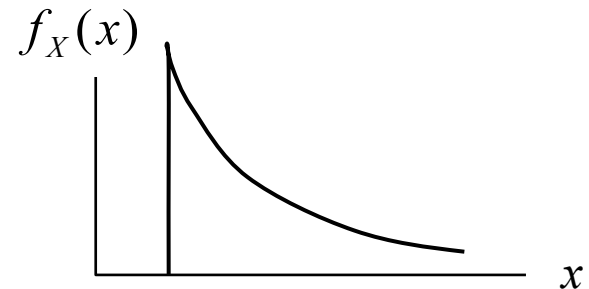
Recall that binomial density function approaches gaussian with mean  $np$  and variance  $np(1-p)$  when  $n$  is large and  $n, p$  satisfy certain conditions.

# Exponential Density and Distribution

- Exponential pdf and distribution function –  
for  $b > 0$  and  $-\infty < a < \infty$ ,

$$f_X(x) = \begin{cases} b^{-1} \exp[-(x-a)/b], & x > a \\ 0, & x < a \end{cases}$$

$$F_X(x) = \begin{cases} 1 - \exp[-(x-a)/b], & x > a \\ 0, & x < a \end{cases}$$



Useful in

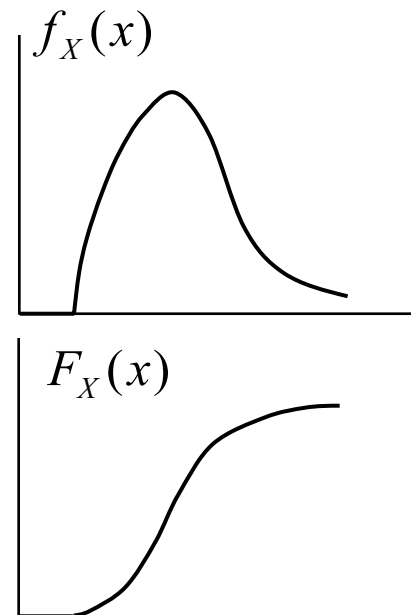
- describing raindrop size when a large number of rainstorm measurements are made
- modeling the fluctuation in strength in received radar signal from a certain types of aircraft
- modeling the interval of arrival time of a certain traffic

# Rayleigh Density and Distribution

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a) \exp\left[\frac{-(x-a)^2}{b}\right], & x \geq a \\ 0, & x < a \end{cases}$$

$$F_X(x) = \begin{cases} 1 - \exp\left[\frac{-(x-a)^2}{b}\right], & x \geq a \\ 0, & x < a \end{cases}$$

$$b > 0, \quad -\infty < a < \infty$$



- Was derived as the density function of the envelop of the sum of many sine waves of different frequencies
- Also used in modeling coordinate errors  $\Delta_r = \sqrt{\Delta_x^2 + \Delta_y^2}$  where  $\Delta_x$  and  $\Delta_y$  are independent gaussian r.v. with zero mean and equal variance.

# Poisson Distribution

$$f_X(x) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \delta(x-k), \quad \lambda > 0$$

Used in describing:

- Number of defective units in a sample taken from a production line
- Number of telephone calls made during a period of time
- Number of customers coming to a store within a period of time
- Number of electrons emitted from a section of cathode in a given time interval

$\lambda$  is the average number of “arrivals”, “calls”, or “emissions” within a unit time interval.



# Other Density & Distribution Functions

- Cauchy  $f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$

$$F_X(x) = \int_{-\infty}^x \frac{dx}{\pi(1+x^2)} = \frac{1}{\pi} \left[ \arctan x + \frac{\pi}{2} \right], \quad -\infty < x < \infty$$

- Mixed Distribution or Mixture

$$f_X(x) = \sum_{k=1}^n c_k p_k(x), \quad \sum_{k=1}^n c_k = 1, \quad \text{and} \quad F_X(x) = \sum_{k=1}^n c_k P_k(x)$$

Where  $p_k(x)$  is some kernel density function such as Gaussian,

$$\text{then, } f_X(x) = \sum_{k=1}^n c_k \mathcal{N}_k(x; \eta_k, \sigma_k^2) \quad \text{and} \quad \bar{X} = \sum_{k=1}^n c_k \eta_k$$

# Functions of Random Variable (Reminder)

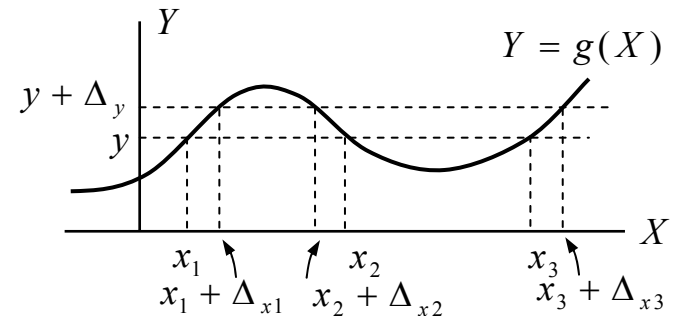
- $X$  is a random variable with pdf  $f_X(x)$ .
- $Y$  is a monotonic function of  $X$ ;  $Y = g(X)$ . Find  $f_Y(y)$ .

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Expressed in  $y$  with  $g^{-1}(y) = x$ ,  $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$

- If  $Y$  is non-monotonic function of  $X$ ,

$$f_Y(y) = \sum_{\text{for all } x=g^{-1}(y)} f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$



Example:  $Y = X^2$  or  $X = \sqrt{Y}$ ,  $\left| \frac{dx}{dy} \right| = \frac{1}{2\sqrt{y}}$ , but for any  $y > 0$ ,  $x = \pm\sqrt{y}$

Therefore,  $f_Y(y) = \frac{1}{2\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})]$ ,  $y \geq 0$ ;  $= 0$ ,  $y < 0$

# Mean Values and Moments

- Mean or expected value of a random variable

$$\bar{X} = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Mean or expected value of a function of a random variable

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

- $N^{\text{th}}$  moments & central moments of a r.v.

$$\overline{X^n} = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

$$\overline{(X - \bar{X})^n} = E[(X - \bar{X})^n] = \int_{-\infty}^{\infty} (x - \bar{X})^n f_X(x) dx$$

When  $n = 2$ ,

$\overline{X^2} = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$  is called the mean - square value

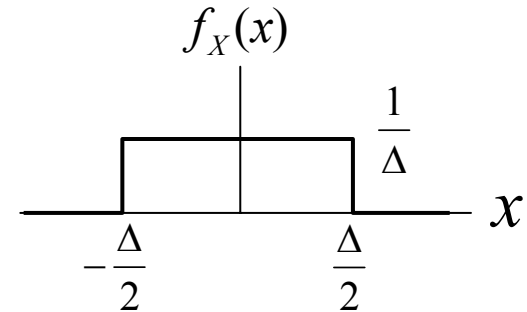
and  $\sigma^2 = E[(X - \bar{X})^2] = \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x) dx$  is the variance.

$$\sigma^2 = E[(X - \bar{X})^2] = E[X^2 - 2X\bar{X} + \bar{X}^2] = E[X^2] - 2\bar{X}^2 + \bar{X}^2 = E[X^2] - \bar{X}^2$$

# Moments of Uniform R.V.

$$f_X(x) = \frac{1}{\Delta} \left[ u\left(x + \frac{\Delta}{2}\right) - u\left(x - \frac{\Delta}{2}\right) \right]$$

$$u(n) = \text{"step function"} = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$\bar{X} = \int_{-\Delta/2}^{\Delta/2} x f_X(x) dx = \int_{-\Delta/2}^{\Delta/2} \frac{x}{\Delta} dx = \frac{x^2}{2\Delta} \Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta}{8} - \frac{\Delta}{8} = 0$$

$$\overline{X^2} = \int_{-\Delta/2}^{\Delta/2} x^2 f_X(x) dx = \int_{-\Delta/2}^{\Delta/2} \frac{x^2}{\Delta} dx = \frac{x^3}{3\Delta} \Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{24} + \frac{\Delta^2}{24} = \frac{\Delta^2}{12}$$

$$\sigma^2 = \overline{X^2} - (\bar{X})^2 = \overline{X^2} = \frac{\Delta^2}{12}$$

$$\overline{X^3} = \int_{-\Delta/2}^{\Delta/2} \frac{x^3}{\Delta} dx = \frac{x^4}{4\Delta} \Big|_{-\Delta/2}^{\Delta/2} = 0$$

The  $n^{\text{th}}$  moment is zero for odd  $n$ .

## Example – Exercise 2-4.2

A random variable  $X$  has a pdf of the form

$$f_X(x) = 0.25[u(x) - u(x - 4)]$$

For the r.v.  $Y=X^2$ , find

a) The mean value

$$\bar{Y} = \int_0^4 x^2 \cdot 0.25 dx = x^3 / 12 \Big|_{x=0}^4 = 16/3$$

b) The mean-square value

$$\overline{Y^2} = \int_0^4 x^4 \cdot 0.25 dx = x^5 / 20 \Big|_{x=0}^4 = 256/5$$

c) The variance.

$$\begin{aligned} \sigma_Y^2 &= \overline{(Y - \bar{Y})^2} = E[Y^2] - \bar{Y}^2 = (256/5) - (16/3)^2 \\ &= 256 \left( \frac{1}{5} - \frac{1}{9} \right) = \frac{1024}{45} \end{aligned}$$

# Entropy – Average Information

- Let  $\{p_i\}_{i=1}^L$  be the (a priori) probability associated with a source which puts out symbols  $\{X = x_i\}_{i=1}^L$  ;
- Define information in symbol  $x_i$  as  $-\log_2 p_i$  ;
- The average information is called entropy and is defined as

$$H = E_X[-\log_2 p_i] = -\sum_{i=1}^L p_i \log_2 p_i$$

For example, if  $L = 2$ , and  $x_1 = 0$ , and  $x_2 = 1$ , then it is a binary source with symbols 0 and 1. And if  $p_1 = p_2 = 0.5$ , then every symbol the source puts out carries one **bit** of information.

if  $p_1 \neq p_2$ , say,  $p_1 = 0.8$  and  $p_2 = 0.2$ , then the average information in each symbol is less than one bit.

# Conditional Probability Distribution

We define the conditional probability the same as before.

$$F_X(x | M) = \Pr(X \leq x | M) = \frac{\Pr(X \leq x, M)}{\Pr(M)}, \quad \Pr(M) > 0$$

If we use the event mapping concept,  $\Pr(X \leq x, M)$  is the probability of all the outcomes which realize both events  $X(\xi) \leq x$  and  $\xi \in M$ .

$$\text{If } M = \{X \leq m\}, \quad F_X(x | M) = \Pr(X \leq x | X \leq m) = \frac{\Pr(X \leq x, X \leq m)}{\Pr(X \leq m)}$$

$$\text{If } x \leq m, \quad F_X(x | M) = \frac{\Pr(X \leq x)}{\Pr(X \leq m)} = \frac{F_X(x)}{F_X(m)} \quad \text{If } x \geq m, \quad F_X(x | M) = \frac{\Pr(X \leq m)}{\Pr(X \leq m)} = 1$$

Conditional probability density function has all the properties of a usual pdf.

$$f_X(x | M) = \frac{dF_X(x | M)}{dx}$$

# Conditional Expected Value

- The expected value of  $X$ , given event  $M$ , is

$$E[X | M] = \int_{-\infty}^{\infty} x f_X(x | M) dx$$

$$\text{if } M = \{X \leq m\}, \quad f_X(x | X \leq m) = \begin{cases} \frac{f_X(x)}{F_X(m)}, & x < m \\ 0, & x \geq m \end{cases}.$$

$$\text{Thus, } E[X | X \leq m] = \frac{\int_{-\infty}^m x f_X(x) dx}{\int_{-\infty}^m f_X(x) dx}$$

which is the expected value of  $X$  when  $X$  is constrained to the set/event  $\{X \leq m\}$ .



# Useful Inequalities

- Chebychev Inequality

$$\Pr\{|X - \bar{X}| \geq \varepsilon\} \leq \sigma_X^2 / \varepsilon^2$$

$$\Pr\{|X - \bar{X}| \geq \varepsilon\} = \int_{-\infty}^{\bar{X}-\varepsilon} f_X(x) dx + \int_{\bar{X}+\varepsilon}^{\infty} f_X(x) dx = \int_{x \in B} f_X(x) dx$$

$$\text{where } B = \{R - \{|x - \bar{X}| < \varepsilon\}\}$$

$$\begin{aligned} \sigma_X^2 &= \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x) dx \geq \int_{x \in B} (x - \bar{X})^2 f_X(x) dx \geq \varepsilon^2 \int_{x \in B} f_X(x) dx \\ &= \varepsilon^2 \Pr\{|X - \bar{X}| \geq \varepsilon\} \end{aligned}$$

- Markov Inequality: for non-negative r.v.  $X$ ,

$$\Pr\{X \geq a\} \leq E[X] / a, \quad a > 0$$