

ECE 3075A
Random Signals

Lecture 6

**Multiple Random Variables, Joint Distributions, and
Correlation**

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Some unfinished topics around single random variables

Entropy – Average Information

- Let $\{p_i\}_{i=1}^L$ be the (a priori) probability associated with a source which puts out symbols $\{X = x_i\}_{i=1}^L$;
- Define information in symbol x_i as $-\log_2 p_i$;
- The average information is called entropy and is defined as

$$H = E_X[-\log_2 p_i] = -\sum_{i=1}^L p_i \log_2 p_i$$

For example, if $L = 2$, and $x_1 = 0$, and $x_2 = 1$, then it is a binary source with symbols 0 and 1. And if $p_1 = p_2 = 0.5$, then every symbol the source puts out carries one **bit** of information.

if $p_1 \neq p_2$, say, $p_1 = 0.8$ and $p_2 = 0.2$, then the average information in each symbol is less than one bit.

Conditional Probability Distribution

We define the conditional probability the same as before.

$$F_X(x | M) = \Pr(X \leq x | M) = \frac{\Pr(X \leq x, M)}{\Pr(M)}, \quad \Pr(M) > 0$$

If we use the event mapping concept, $\Pr(X \leq x, M)$ is the probability of all the outcomes which realize both events $X(\xi) \leq x$ and $\xi \in M$.

$$\text{If } M = \{X \leq m\}, \quad F_X(x | M) = \Pr(X \leq x | X \leq m) = \frac{\Pr(X \leq x, X \leq m)}{\Pr(X \leq m)}$$

$$\text{If } x \leq m, \quad F_X(x | M) = \frac{\Pr(X \leq x)}{\Pr(X \leq m)} = \frac{F_X(x)}{F_X(m)} \quad \text{If } x \geq m, \quad F_X(x | M) = \frac{\Pr(X \leq m)}{\Pr(X \leq m)} = 1$$

Conditional probability density function has all the properties of a usual pdf.

$$f_X(x | M) = \frac{dF_X(x | M)}{dx}$$

Conditional Expected Value

- The expected value of X , given event M , is

$$E[X | M] = \int_{-\infty}^{\infty} x f_X(x | M) dx$$

$$\text{if } M = \{X \leq m\}, \quad f_X(x | X \leq m) = \begin{cases} \frac{f_X(x)}{F_X(m)}, & x < m \\ 0, & x \geq m \end{cases}.$$

$$\text{Thus, } E[X | X \leq m] = \frac{\int_{-\infty}^m x f_X(x) dx}{\int_{-\infty}^m f_X(x) dx}$$

which is the expected value of X when X is constrained to the set/event $\{X \leq m\}$.

Useful & Intriguing Inequalities

- Chebychev Inequality

$$\Pr\{|X - \bar{X}| \geq \varepsilon\} \leq \sigma_X^2 / \varepsilon^2$$

$$\Pr\{|X - \bar{X}| \geq \varepsilon\} = \int_{-\infty}^{\bar{X}-\varepsilon} f_X(x) dx + \int_{\bar{X}+\varepsilon}^{\infty} f_X(x) dx = \int_{x \in B} f_X(x) dx$$

$$\text{where } B = \{R - \{|x - \bar{X}| < \varepsilon\}\}$$

$$\begin{aligned} \sigma_X^2 &= \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x) dx \geq \int_{x \in B} (x - \bar{X})^2 f_X(x) dx \geq \varepsilon^2 \int_{x \in B} f_X(x) dx \\ &= \varepsilon^2 \Pr\{|X - \bar{X}| \geq \varepsilon\} \end{aligned}$$

- Markov Inequality: for non-negative r.v. X ,

$$\Pr\{X \geq a\} \leq E[X] / a, \quad a > 0$$

Monte Carlo Simulation

- To generate a random variable with prescribed probability distribution function, often needed in computer simulation of a certain physical phenomenon such as noise in a measurement;
- Construction of such a simulation begins with a random variable that has uniform distribution on $[0,1]$;
- Find a monotonically non-decreasing transformation $Y=T(X)$ such that Y has the prescribed distribution $F_Y(y)$.

$$\Pr\{Y \leq y\} = F_Y[y = T(x)] = F_X(x) = \Pr\{X \leq x\}$$

Since X is uniformly distributed, $F_X(x) = x$ for $0 \leq x \leq 1$.

Transformation $y = T(x) = F_Y^{-1}(x)$, for $0 \leq x \leq 1$ will produce the desired simulation of r.v. Y .

Uniform Random Number Generator

- Algorithmic generation of *random* numbers that display uniform distribution between 0 and 1;
- No algorithm can produce true random numbers; they are called pseudo-random numbers instead;
- Integer linear congruence method is often used:

$$x_{n+1} = ((ax_n + c))_m = (ax_n + c)_{\text{mod } m}$$

where a and c are certain fixed values and x_0 is an initial number known as the seed. Normalizing the result by m will produce a pseudo-random number sequence with value in $[0, 1)$. Such a sequence is periodic with a “cycle” less than or equal to m .

Example of Monte Carlo Simulation

Let $a = 9$, $c = 5$, $m = 16$, and $x_0 = 4$.

$$x_{n+1} = (9x_n + 5)_{\text{mod } 16}$$

Try $a = 9$, $c = 3$, $m = 16$, and $x_0 = 4$,
and see what happens.

Integer sequence: 4, 9, 6, 11, 8, 13, 10, 15, 12, 1, 14, 3, 0, 5, 2, 7, 4,

Normalized plus an epsilon (0.0001):

0.2501, 0.5626, 0.3751, 0.6876, 0.5001, 0.8126, 0.6251, 0.9376, 0.7501, 0.0626,
0.8751, 0.1876, 0, 0.3126, 0.1251, 0.4376, 0.2501, ...

E.g., Desired Probability Distribution Function: $F_Y(y) = \frac{1}{1 + e^{-ay}}$

$$F_Y(y) = \frac{1}{1 + e^{-ay}} = x, \quad y = -\frac{1}{a} \ln\left(\frac{1}{x} - 1\right)$$

$y = -0.54904, 0.12586, -0.2552, 0.394461, 0.0002, 0.733497, 0.255626, 1.354879, 0.549573,$
 $-1.35317, 0.973412, -0.73284, -4.60512, -0.394, -0.9725, -0.12545, -0.54904, \dots$

What is the corresponding probability density function $f_Y(y)$?

Another Example of RNG

$$s = 2111111111x_{n-4} + 1492x_{n-3} + 1776x_{n-2} + 5115x_{n-1} + c$$

a 64-bit unsigned integer

$$x_n = (s)_{\text{mod } 2^{32}}$$

$$\text{where } c = \left\lfloor \frac{s}{2^{32}} \right\rfloor = \text{largest integer less than or equal to } \frac{s}{2^{32}}$$

The s in calculating the floor above is the previous value (during calculation of x_{n-1}).

Specifications:

- 32-bit integer output
- cycle length is $3 \cdot 10^{47}$

Characteristic Function

- Definition

$$\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{jux} f_X(x) dx$$

that is, the characteristic function of a random variable can be viewed as the Fourier transform of its probability density function.

- The pdf is then the inverse Fourier transform of the characteristic function

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jux} \Phi_X(u) du$$

$$\frac{d}{du} \Phi(u) = \int_{-\infty}^{\infty} \left(\frac{d}{du} e^{jux} \right) f_X(x) dx = \int_{-\infty}^{\infty} jx e^{jux} f_X(x) dx$$

$$\left. \frac{d}{du} \Phi(u) \right|_{u=0} = \int_{-\infty}^{\infty} jx f_X(x) dx = j\bar{X}$$

$$\left. \frac{d^n}{du^n} \Phi(u) \right|_{u=0} = j^n E[X^n] = j^n \overline{X^n}$$

Main Subjects of This Lecture

Two Random Variables

- As in combined experiments, we are interested in the co-occurrence of separate events.

Try to recall and answer: Do they have to be simultaneous? What is the difference, if any, between tossing a die and a coin together and tossing them separately? What is a combined event in the Cartesian product observation/sample space? What is the definition of **independent** experiments? And, likewise, **dependent** experiment?

- **Joint Probability Distribution Function** of two r.v., X and Y

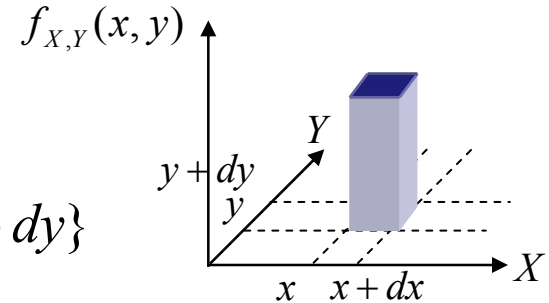
$$F_{X,Y}(x, y) = \Pr\{X \leq x, Y \leq y\}$$

1. $0 \leq F_{X,Y}(x, y) \leq 1, \quad -\infty < x < \infty, -\infty < y < \infty$
2. $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = F_{X,Y}(-\infty, -\infty) = 0$
3. $F_{X,Y}(\infty, \infty) = 1$
4. $F_{X,Y}(x, y)$ is a non-decreasing function of either x or y
5. $F_{X,Y}(\infty, y) = F_Y(y), \quad F_{X,Y}(x, \infty) = F_X(x)$

Joint Probability Density Functions

Definition $f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$

$$f_{X,Y}(x,y) dx dy = \Pr\{x < X \leq x + dx, y < Y \leq y + dy\}$$



1. $f_{X,Y}(x,y) \geq 0, \quad -\infty < x < \infty, -\infty < y < \infty$

2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$

3. $F_{X,Y}(x,y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(u,v) dv du$

4. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx \quad \rightarrow$ marginals

5. $\Pr\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{X,Y}(x,y) dy dx$

Example

The joint density function of X and Y is given by

$$f_{X,Y}(x, y) = u(x)u(y)xe^{-x(y+1)}$$

Find the marginal density functions.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy = \int_0^{\infty} u(x)xe^{-x(y+1)} dy \\ &= u(x)xe^{-x} \int_0^{\infty} e^{-xy} dy = u(x)xe^{-x} \left. \frac{e^{-xy}}{-x} \right|_{y=0}^{\infty} = u(x)e^{-x} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx = \int_0^{\infty} u(y)xe^{-x(y+1)} dx \\ &= u(y)e^{-x(y+1)} \left[\frac{x}{-(y+1)} - \frac{1}{(y+1)^2} \right] \Big|_{x=0}^{\infty} = \frac{u(y)}{(y+1)^2} \end{aligned}$$

Expectation of Functions of Two R.V.s

- Similar definition as in single random variable case

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X, Y}(x, y) dx dy$$

- When $g(X, Y) = X^n Y^k$

$E[X^n Y^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^n y^k f_{X, Y}(x, y) dx dy$ is called the **joint moment of X and Y**. When $n=k=1$, it is called correlation. $n+k$ is the order of the moment.

$$\begin{aligned} E[(X - \bar{X})(Y - \bar{Y})] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{X})(y - \bar{Y}) f_{X, Y}(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (xy - \bar{X}y - x\bar{Y} + \bar{X}\bar{Y}) f_{X, Y}(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X, Y}(x, y) dx dy - \bar{X}\bar{Y} = \overline{XY} - \bar{X}\bar{Y} \end{aligned}$$

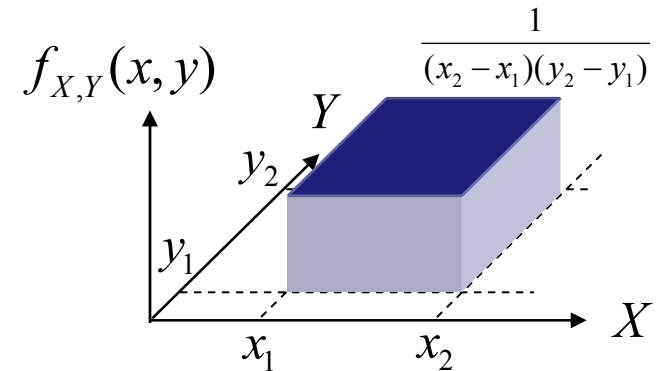
Two Uniform Random Variables

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{(x_2 - x_1)(y_2 - y_1)}, & x_1 < x \leq x_2, y_1 < y \leq y_2 \\ 0, & \text{elsewhere} \end{cases}$$

Marginals

$$f_X(x) = \int_{y_1}^{y_2} \frac{1}{(x_2 - x_1)(y_2 - y_1)} dy = \frac{y}{(x_2 - x_1)(y_2 - y_1)} \Big|_{y_1}^{y_2} = \frac{1}{x_2 - x_1}$$

$$f_Y(y) = \int_{x_1}^{x_2} \frac{1}{(x_2 - x_1)(y_2 - y_1)} dx = \frac{x}{(x_2 - x_1)(y_2 - y_1)} \Big|_{x_1}^{x_2} = \frac{1}{y_2 - y_1}$$



$$E[XY] = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{xy}{(x_2 - x_1)(y_2 - y_1)} dx dy = \frac{1}{(x_2 - x_1)(y_2 - y_1)} \left[\frac{x^2}{2} \Big|_{x_1}^{x_2} \right] \left[\frac{y^2}{2} \Big|_{y_1}^{y_2} \right] = \frac{(x_1 + x_2)(y_1 + y_2)}{4}$$

$$F_{X,Y}(x, y) = \int_{x_1}^x \int_{y_1}^y \frac{1}{(x_2 - x_1)(y_2 - y_1)} dv du = \frac{(x - x_1)(y - y_1)}{(x_2 - x_1)(y_2 - y_1)}$$