ECE 3075A
Random Signals

Lecture 7
More on Joint Distributions, Joint Moments, Marginal Distributions and Statistical Independence

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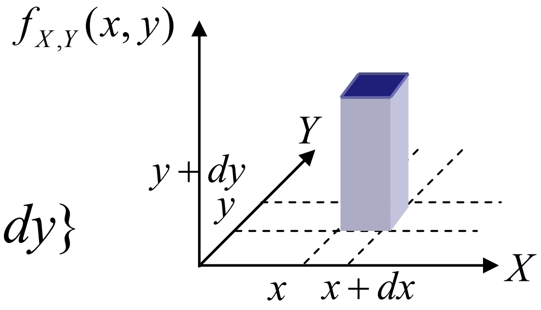
Joint Probability Density Functions

Definition $f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$



$$f_{X,Y}(x,y)\,dx\,dy = \Pr\{x < X \le x+dx, y < Y \le y+dy\}$$

1. $f_{X,Y}(x,y) \ge 0, \quad -\infty < x < \infty, -\infty < y < \infty$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$
3. $F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,dv\,du$
4. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$ $\to$ marginals
5. $\Pr\{x_1 < X \le x_2, y_1 < Y \le y_2\} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dy\,dx$

Example

The joint density function of X and Y is given by

$$f_{X,Y}(x,y) = u(x)u(y)xe^{-x(y+1)}$$

Find the marginal density functions.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{\infty} u(x)xe^{-x(y+1)}dy$$

$$= u(x)xe^{-x}\int_0^{\infty} e^{-xy}dy = u(x)xe^{-x}\left.\frac{e^{-xy}}{-x}\right|_{y=0}^{\infty} = u(x)e^{-x}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^{\infty} u(y)xe^{-x(y+1)}dx$$

$$= u(y)e^{-x(y+1)}\left[\frac{x}{-(y+1)} - \frac{1}{(y+1)^2}\right]\Bigg|_{x=0}^{\infty} = \frac{u(y)}{(y+1)^2}$$

note : $xe^{-ax} = \frac{1}{a}\frac{d}{dx}\left(-\frac{1}{a}e^{-ax} - xe^{-ax}\right)$

Expectation of Functions of Two R.V.s

- Similar definition as in single random variable case

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

- When $g(X,Y) = X^n Y^k$

$$E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k\, f_{X,Y}(x,y)\, dx\, dy$$ is called the ***joint moment of*** X ***and*** Y. When $n=k=1$, it is called correlation. $n+k$ is the order of the moment.

$$E[(X-\overline{X})(Y-\overline{Y})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\overline{X})(y-\overline{Y})\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy - \overline{X}y - x\overline{Y} + \overline{X}\,\overline{Y})\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\, dx\, dy - \overline{X}\,\overline{Y} = \overline{XY} - \overline{X}\,\overline{Y}$$

Joint Central Moments

$$E\left[(X-\overline{X})^n (Y-\overline{Y})^k\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\overline{X})^n (y-\overline{Y})^k \, f_{X,Y}(x,y)\, dx\, dy = \mu_{nk}$$

$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy = R_{XY}$ is the correlation between X and Y

If $R_{XY} = E[XY] = E[X]E[Y]$, then X and Y are said to be uncorrelated.

If $R_{XY} = 0$, then X and Y are said to be orthogonal.

μ_{11} is called covariance.

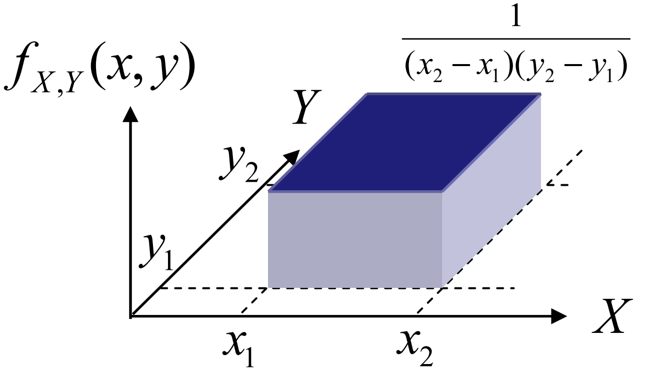
$$\rho = E\left\{\left[\frac{X-\overline{X}}{\sigma_X}\right]\left[\frac{Y-\overline{Y}}{\sigma_Y}\right]\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x-\overline{X}}{\sigma_X}\,\frac{y-\overline{Y}}{\sigma_Y}\, f_{X,Y}(x,y)\, dx\, dy$$

ρ is called the **correlation coefficient** or **normalized covariance**, which expresses the degree of correlation between the two r.v.s without regard to the magnitude of either one r.v.

Two Uniform Random Variables

$$f_{X,Y}(x,y)=\begin{cases}\dfrac{1}{(x_2-x_1)(y_2-y_1)}, & x_1<x\le x_2,\ y_1<y\le y_2\\ 0, & \text{elsewhere}\end{cases}$$

Marginals



$$f_X(x)=\int_{y_1}^{y_2}\frac{1}{(x_2-x_1)(y_2-y_1)}dy=\frac{y}{(x_2-x_1)(y_2-y_1)}\Bigg|_{y_1}^{y_2}=\frac{1}{x_2-x_1}$$

$$f_Y(y)=\int_{x_1}^{x_2}\frac{1}{(x_2-x_1)(y_2-y_1)}dx=\frac{x}{(x_2-x_1)(y_2-y_1)}\Bigg|_{x_1}^{x_2}=\frac{1}{y_2-y_1}$$

$$E[XY]=\int_{y_1}^{y_2}\int_{x_1}^{x_2}\frac{xy}{(x_2-x_1)(y_2-y_1)}dx\,dy=\frac{1}{(x_2-x_1)(y_2-y_1)}\left[\frac{x^2}{2}\Bigg|_{x_1}^{x_2}\right]\left[\frac{y^2}{2}\Bigg|_{y_1}^{y_2}\right]=\frac{(x_1+x_2)(y_1+y_2)}{4}$$

$$F_{X,Y}(x,y)=\int_{x_1}^{x}\int_{y_1}^{y}\frac{1}{(x_2-x_1)(y_2-y_1)}dv\,du=\frac{(x-x_1)(y-y_1)}{(x_2-x_1)(y_2-y_1)}$$

More on Correlation Coefficient

$$\rho = E\left\{\left[\frac{X-\overline{X}}{\sigma_X}\right]\left[\frac{Y-\overline{Y}}{\sigma_Y}\right]\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{x-\overline{X}}{\sigma_X}\right]\left[\frac{y-\overline{Y}}{\sigma_Y}\right] f_{XY}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{xy-\overline{X}y-x\overline{Y}+\overline{X}\overline{Y}}{\sigma_X\sigma_Y} f_{XY}(x,y)\,dx\,dy = \frac{E[XY]-\overline{X}\overline{Y}}{\sigma_X\sigma_Y}$$

Define normalized r.v. $\xi = \dfrac{X-\overline{X}}{\sigma_X}$ and $\eta = \dfrac{Y-\overline{Y}}{\sigma_Y}$

Both ξ and η are zero mean and unit variance, that is

$\overline{\xi} = \overline{\eta} = 0$ and $\sigma_\xi^2 = \sigma_\eta^2 = 1$. Then, $\rho = E[\xi\eta]$

$$E[(\xi \pm \eta)^2] = E[\xi^2 \pm 2\xi\eta + \eta^2] = \sigma_\xi^2 \pm 2\rho + \sigma_\eta^2 = 2(1 \pm \rho) \geq 0$$

Therefore, $-1 \leq \rho \leq 1$

And if the two r.v.s are independent, $\rho = E[\xi\eta] = \overline{\xi}\,\overline{\eta} = 0$

Example – Exercise 3-4.1

Two random variables have means of 1 and variances of 1 and 4, respectively. Their correlation coefficient is 0.5.

a) Find the variance of their sum.

b) Find the mean square value of their sum.

c) Find the mean square value of their difference.

$$1 = \sigma_X^2 = \overline{X^2} - \overline{X}^2 = \overline{X^2} - 1 \quad \therefore \overline{X^2} = 2, \text{ and } 4 = \sigma_Y^2 = \overline{Y^2} - \overline{Y}^2 = \overline{Y^2} - 1 \quad \therefore \overline{Y^2} = 5$$

$$0.5 = \rho = \frac{E[XY] - \overline{X}\,\overline{Y}}{\sigma_X \sigma_Y} = \frac{\overline{XY} - 1}{2} \quad \therefore \overline{XY} = 2$$

$$\sigma_{X+Y}^2 = E[(X + Y - \overline{X} - \overline{Y})^2] = \overline{X^2} + 2\overline{XY} + \overline{Y^2} - (\overline{X} + \overline{Y})^2$$

$$= 2 + 4 + 5 - 4 = 7$$

(Is this independent of the means?)

$$E[(X+Y)^2] = \overline{X^2} + 2\overline{XY} + \overline{Y^2} = 2 + 4 + 5 = 11$$

$$E[(X-Y)^2] = \overline{X^2} - 2\overline{XY} + \overline{Y^2} = 2 - 4 + 5 = 3$$

Conditional Probability

Recall $F_X(x \mid M) = \Pr(X \le x \mid M) = \dfrac{\Pr(X \le x, M)}{\Pr(M)}, \quad \Pr(M) > 0$

If $M = \{Y \le y\}$,

$$F_X(x \mid M) = \Pr(X \le x \mid Y \le y) = \frac{\Pr(X \le x, Y \le y)}{\Pr(Y \le y)} = \frac{F_{X,Y}(x, y)}{F_Y(y)}$$

Similarly, $F_X(x \mid y_1 < Y \le y_2) = \dfrac{F_{X,Y}(x, y_2) - F_{X,Y}(x, y_1)}{F_Y(y_2) - F_Y(y_1)}$

$$F_X(x \mid Y = y) = \lim_{\Delta y \to 0} \frac{F_{X,Y}(x, y + \Delta y) - F_{X,Y}(x, y)}{F_Y(y + \Delta y) - F_Y(y)} = \frac{\partial F_{X,Y}(x, y) / \partial y}{\partial F_Y(y) / \partial y} = \frac{\int_{-\infty}^{x} f_{X,Y}(u, y)\, du}{f_Y(y)}$$

$$f_X(x \mid Y = y) = \frac{\frac{d}{dx} \int_{-\infty}^{x} f_{X,Y}(u, y)\, du}{f_Y(y)} = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \qquad f_Y(y \mid X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

Notation issue coming up!

Conditional Probability - Notation

$$f_{Y|X}(y \mid X = x) = f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

$$f_{X|Y}(x \mid Y = y) = f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

We use $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$ to make explicit the fact that x and y are "*realized*" **random variables**, rather than a parameter that define the density function $f(y \mid x)$ or $f(x \mid y)$. If they are "parameters", we'd use the notation like $f(y ; x)$ or $f(x ; y)$ [or more likely, $f(y;\, a)$ or $f(x;\, b)$]

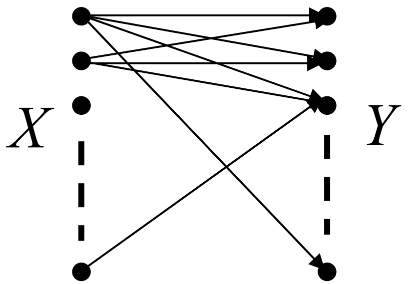
Marginals:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = \int_{-\infty}^{\infty} f_{X|Y}(x \mid y) f_Y(y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx = \int_{-\infty}^{\infty} f_{Y|X}(y \mid x) f_X(x)\, dx$$

Use of Conditional Probability - Example

- Consider an additive noise channel $Y = X + N$. I.e., a "random" source puts out signal X but at the destination, it is observed as Y due to additive noise N.
- We are often interested in estimating x (i.e., what was sent) based on the received signal y.
- Use Bayes formula



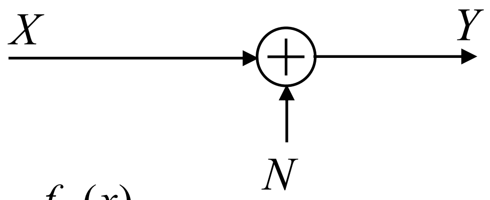
$$f_{X|Y}(x\,|\,y) = \frac{f_{Y|X}(y\,|\,x) f_X(x)}{f_Y(y)}$$

$$f_{Y|X}(y\,|\,x) = f_N(n) = f_N(y - x)$$ [Note: here, we are looking at $Y = x+N$, because x is known]

$$f_{X|Y}(x\,|\,y) = \frac{f_N(y - x) f_X(x)}{f_Y(y)} = \frac{f_N(y - x) f_X(x)}{\int_{-\infty}^{\infty} f_N(y - x) f_X(x)\, dx}$$

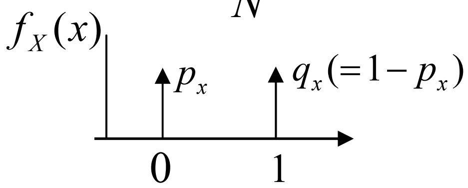
For a given observation y, $\arg\max_x f_{X|Y}(x\,|\,y)$ is a good estimate of the original symbol.

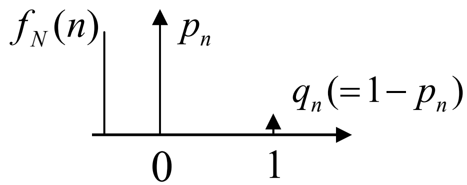
Binary Channel Example

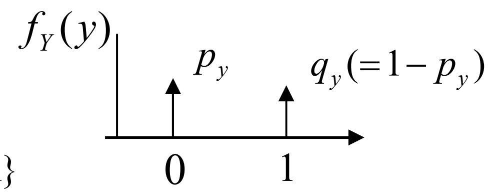


$$Y = X + N$$

X	1	1	0	0
N	0	1	0	1
Y	1	0	0	1







$$p_y = \Pr\{Y=0\} = \Pr\{X=0\}\Pr\{Y=0 \mid X=0\} + \Pr\{X=1\}\Pr\{Y=0 \mid X=1\}$$

$$\Pr\{Y=0 \mid X=0\} = \Pr\{N=0\} = p_n, \quad \Pr\{Y=0 \mid X=1\} = \Pr\{N=1\} = q_n$$

$\therefore \; p_y = p_x p_n + q_x q_n = p_x p_n + (1-p_x)(1-p_n) = 1 - p_x - p_n + 2p_x p_n$ and $q_y = 1 - p_y = p_x + p_n - 2p_x p_n$

$$f_{X|Y}(x \mid y) = f_{Y|X}(y \mid x) f_X(x) [f_Y(y)]^{-1} = f_N(y-x) f_X(x) \left[\int_{-\infty}^{\infty} f_N(y-x) f_X(x)\, dx \right]^{-1}$$

y	x	$f_{X\|Y}(x \mid y)$
0	0	$p_n p_x (p_y)^{-1}$
0	1	$q_n q_x (p_y)^{-1}$
1	0	$q_n p_x (q_y)^{-1}$
1	1	$p_n q_x (q_y)^{-1}$

Let $p_x = 0.5$ and $p_n = 0.9$.

$\rightarrow p_y = 0.5 * 0.9 + 0.5 * 0.1 = 0.5$

y	x	$f_{X\|Y}(x \mid y)$
0	0	**0.9** ←
0	1	0.1
1	0	0.1
1	1	**0.9** ←

Let $p_x = 0.3$ and $p_n = 0.6$.

$\rightarrow p_y = 0.3 * 0.6 + 0.7 * 0.4 = 0.46$

y	x	$f_{X\|Y}(x \mid y)$
0	0	9/23
0	1	**14/23** ←
1	0	2/9
1	1	**7/9** ←

Modeling Measurement & Noise - Example

- In space probe, high energy particles are received from space. The interval between particle arrival times is considered a measurement of a certain space activity. (Recall Poisson distribution.) The arrival interval is a random signal with exponential distribution

$$f_X(x) = \begin{matrix} b\exp(-bx) & x \geq 0 \\ 0 & x < 0 \end{matrix}$$

- The measured signal usually has an additive noise component with Gaussian distribution:

$$Y = X + N \quad \text{and} \quad f_N(n) = (\sqrt{2\pi}\sigma_N)^{-1}\exp(-n^2/2\sigma_N^2)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)\,dx = \int_{-\infty}^{\infty} f_N(y-x) f_X(x)\,dx$$

$$= \int_0^{\infty} \frac{b}{\sqrt{2\pi}\sigma_N}\exp(-bx)\exp\left(-\frac{(y-x)^2}{2\sigma_N^2}\right)dx = b\exp\left(-by + \frac{b^2\sigma_N^2}{2}\right)Q\left(-\frac{y-b\sigma_N^2}{\sigma_N}\right)$$

Measurement & Noise – Example (cont'd)

- The a posteriori probability density function is

$$f_{X|Y}(x \mid y) = \begin{cases} [f_Y(y)]^{-1} \dfrac{b}{\sqrt{2\pi}\sigma_N} \exp(-bx) \exp\left(-\dfrac{(y-x)^2}{2\sigma_N^2}\right), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$f_{X|Y}(x \mid y) = \begin{cases} [f_Y(y)]^{-1} \dfrac{b}{\sqrt{2\pi}\sigma_N} \exp\left(-bx - \dfrac{(y-x)^2}{2\sigma_N^2}\right), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Given y (received, known), we estimate the signal x (unknown) by

$\arg\max_x f_{X|Y}(x \mid y)$ which is achieved at $\arg\min_x 2\sigma_N^2 bx + (y-x)^2$

$$\frac{d}{dx} 2\sigma_N^2 bx + (y-x)^2 = 2\sigma_N^2 b - 2(y-x) = 0, \quad \rightarrow \quad x = y - \sigma_N^2 b \text{ if } y \geq \sigma_N^2 b$$

Therefore, if $y \geq \sigma_N^2 b$, $\hat{x} = y - \sigma_N^2 b$, and $y < \sigma_N^2 b$, $\hat{x} = 0$

Statistical Independence

- When two random variables are independent, a knowledge of one r.v. gives no information about the value of the other.

X and Y are statistically independent, *iff* $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} f_X(x)\,dx \int_{-\infty}^{\infty} f_Y(y)\,dy = E[X]E[Y]$$

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = f_X(x), \quad f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = f_Y(y)$$

Another way to see independence – if the joint probability density function can be factored into product of a function of x only and a function of y only, then the two r.v.s are statistically independent.

E.g., if $f_{X,Y}(x,y) = u(x)u(y)xe^{-x(y+1)}$, can X and Y be independent?

Exercise 3-3.1

Two random variables, X and Y, have a joint probability density function of the form

$$f_{X,Y}(x,y) = \begin{cases} ke^{-(x+ay-1)}, & 0 \le x \le \infty, 1 \le y \le \infty \\ 0, & \text{elsewhere} \end{cases}$$

Find
a) values of k and a for which the random variables are statistically independent; b) the expected value of XY.

Since the joint pdf is separable, the two r.v.s are independent. We need to find a and k such that the function is a legitimate joint pdf.

$$\int_0^\infty \int_1^\infty f_{X,Y}(x,y)\,dy\,dx = \int_0^\infty \int_1^\infty ke^{-(x+ay-1)}\,dy\,dx = ke\int_0^\infty e^{-x}dx\int_1^\infty e^{-ay}\,dy$$

$$= ke\left[-e^{-x}\right]_{x=0}^{\infty}\left[-\frac{e^{-ay}}{a}\right]_{y=1}^{\infty} = ke \bullet 1 \bullet \frac{e^{-a}}{a} = 1, \quad \rightarrow \; a = 1, \text{ and } \; k = 1$$

$$E[XY] = e\int_0^\infty xe^{-x}dx\int_1^\infty ye^{-y}dy = e \bullet 1 \bullet 2e^{-1} = 2$$

More on Statistical Independence

- Random variables $X_1, X_2, \cdots, X_N$ are independent,

  $$\textit{iff} \quad \Pr\{X_1 \le x_1, X_2 \le x_2, \cdots, X_N \le x_N\}$$
  $$= \Pr\{X_1 \le x_1\}\Pr\{X_2 \le x_2\}\cdots\Pr\{X_N \le x_N\}$$

  Similarly, they are independent *iff*

  $$f_{X_1, X_2, \cdots, X_N}(x_1, x_2, \cdots, x_N) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_N}(x_N)$$

- If $X_1, X_2, \cdots, X_N$ are independent, then any sub-group of the random variables are independent. (Why?)
- However, generalization is not guaranteed: e.g., pair-wise independence does not immediately imply overall independence.