ECE 3075A

Random Signals

Lecture 8

Distributions of Functions of Several Random Variables

School of Electrical and Computer Engineering
Georgia Institute of Technology
Summer, 2003

Statistical Independence

- When two random variables are independent, a knowledge of one r.v. gives no information about the value of the other.

X and Y are statistically independent, $iff\ f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} f_X(x)\,dx \int_{-\infty}^{\infty} f_Y(y)\,dy = E[X]E[Y]$$

$$f_{X|Y}(x\,|\,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = f_X(x), \quad f_{Y|X}(y\,|\,x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = f_Y(y)$$

Another way to see independence – if the joint probability density function can be factored into product of a function of x only and a function of y only, then the two r.v.s are statistically independent.

E.g., if $f_{X,Y}(x,y) = u(x)u(y)xe^{-x(y+1)}$, can X and Y be independent?

Exercise 3-3.1

Two random variables, X and Y, have a joint probability density function of the form

$$f_{X,Y}(x,y) = \begin{cases} ke^{-(x+ay-1)}, & 0 \le x \le \infty, 1 \le y \le \infty \\ 0, & \text{elsewhere} \end{cases}$$

Find

a) values of k and a for which the random variables are statistically independent; b) the expected value of XY.

Since the joint pdf is separable, the two r.v.s are independent. We need to find a and k such that the function is a legitimate joint pdf.

$$\int_0^\infty \int_1^\infty f_{X,Y}(x,y)\,dy\,dx = \int_0^\infty \int_1^\infty ke^{-(x+ay-1)}\,dy\,dx = ke\int_0^\infty e^{-x}dx\int_1^\infty e^{-ay}\,dy$$

$$= ke\left[-e^{-x}\right]_{x=0}^{\infty}\left[-\frac{e^{-ay}}{a}\right]_{y=1}^{\infty} = ke \bullet 1 \bullet \frac{e^{-a}}{a} = 1, \quad \rightarrow \; a = 1, \text{ and } k = 1$$

$$E[XY] = e\int_0^\infty xe^{-x}dx\int_1^\infty ye^{-y}dy = e \bullet 1 \bullet 2e^{-1} = 2$$

More on Statistical Independence

- Random variables $X_1, X_2, \cdots, X_N$ are independent,

$$\textit{iff} \quad \Pr\{X_1 \le x_1, X_2 \le x_2, \cdots, X_N \le x_N\}$$
$$= \Pr\{X_1 \le x_1\}\Pr\{X_2 \le x_2\}\cdots\Pr\{X_N \le x_N\}$$

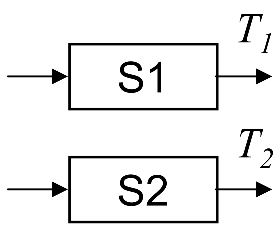
Similarly, they are independent *iff*

$$f_{X_1,X_2,\cdots,X_N}(x_1, x_2, \cdots, x_N) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_N}(x_N)$$

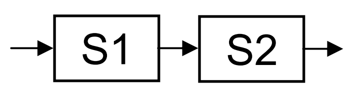
- If $X_1, X_2, \cdots, X_N$ are independent, then any sub-group of the random variables are independent. (Why?)
- However, generalization is not guaranteed: e.g., pair-wise independence does not immediately imply overall independence.

Probability Distributions of Functions of Several Random Variables

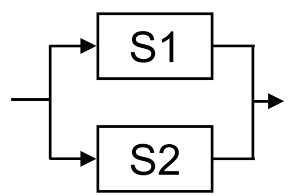
- Observation data usually can be modeled by certain distributions – we have an extensive set of distribution functions to work with (from normal distribution to mixture).
- We study the joint distribution of several random events to understand and make use of the statistical relationship among them. For example, the noise level and the bit error rate in digital communication.
- Yet often we are interested in functions of these random events, to construct new knowledge or for new use that is not obvious from the random events themselves.



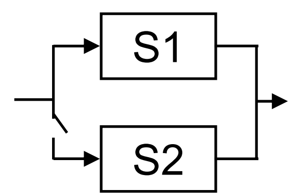
Interval betw'n onset of use and failure is a r.v.



$T = \min(T_1, T_2)$



$T = \max(T_1, T_2)$



$T = T_1 + T_2$

Probability Distributions of Functions of Several Random Variables

- The probability space is defined on a hyperspace that contains the random variables.
- Function $Y = g(X_1, X_2, \cdots, X_N)$

$$F_Y(y) = \Pr(Y \le y) = \Pr[g(X_1, X_2, \cdots, X_N) \le y]$$
$$= \Pr\left(\{\xi : g(x_1 = X_1(\xi), x_2 = X_2(\xi), \cdots, x_N = X_N(\xi)) \le y\}\right)$$

$$F_Y(y) = \Pr[g(X_1, X_2, \cdots, X_N) \le y] =$$
$$\int \cdots \int_{\{g(x_1, x_2, \cdots, x_N) \le y\}} f_{X_1, X_2, \cdots, X_N}(x_1, x_2, \cdots, x_N) dx_1 dx_2 \cdots dx_N$$

$\Longrightarrow$ Direct integration

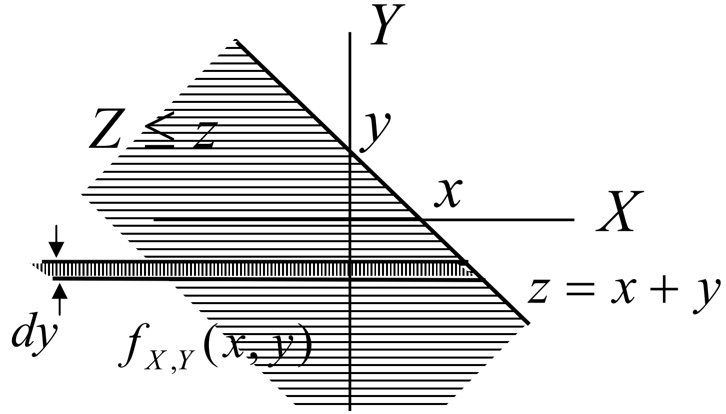
- Another method is through transformation of variables.

Sum of Two Random Variables

$$Z = X + Y$$

At any y, the small stripe has a probability mass of

$$dy\int_{-\infty}^{z-y} f_{X,Y}(x,y)dx$$



Thus, the shaded area which is $\Pr\{Z \le z\}$ can be obtained by

$$\Pr\{Z \le z\} = F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{X,Y}(x,y)\,dx\,dy$$

If X and Y are independent, $F_Z(z) = \int_{-\infty}^{\infty} f_Y(y)\int_{-\infty}^{z-y} f_X(x)\,dx\,dy$

And, $$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z-y)\,dy$$

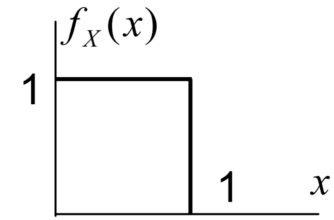
Use Leibniz's rule

The pdf of the sum of two statistically independent random variables is the convolution of their individual pdf's.

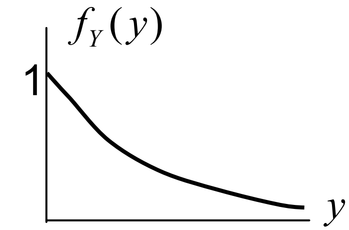
Sum of Random Variables - Example

The pdf of X and Y, which are independent, are

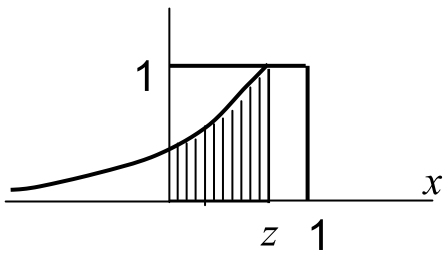
$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

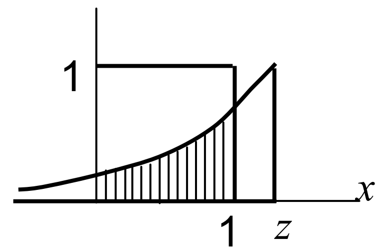


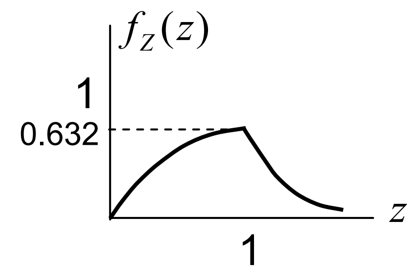
$$f_Y(y) = \begin{cases} e^{-y}, & y \ge 0 \\ 0, & y < 0 \end{cases}$$



$$Z = X + Y$$







$0 < z \le 1$

$$f_Z(z) = \int_0^1 e^{-(z-x)} u(z-x)dx = \int_0^z e^{-(z-x)} dx = e^{-z}[e^x]_{x=0}^z = 1 - e^{-z}$$

$1 < z < \infty$

$$f_Z(z) = \int_0^1 e^{-(z-x)} u(z-x)dx = \int_0^1 e^{-(z-x)} dx = e^{-z}[e^x]_{x=0}^1 = e^{-z}(e-1)$$

Probability Density Functions of Two R.V.s

$$Z = \min(X, Y)$$

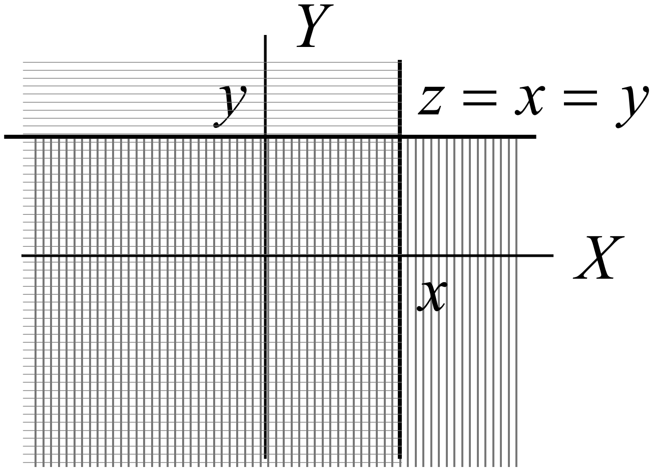
$$\{Z \le z\} = \{X \le z\} \cup \{Y \le z\}$$

$$\Pr\{Z \le z\} = \Pr\{X \le z\} + \Pr\{Y \le z\}$$
$$- \Pr\{X \le z, Y \le z\}$$

$$F_Z(z) = F_X(z) + F_Y(z) - F_{X,Y}(z,z)$$

If X and Y are independent,

$$f_Z(z) = f_X(z) + f_Y(z) - f_X(z)F_Y(z) - F_X(z)f_Y(z)$$
$$= f_X(z)[1 - F_Y(z)] + f_Y(z)[1 - F_X(z)]$$



$$Z = \max(X, Y)$$

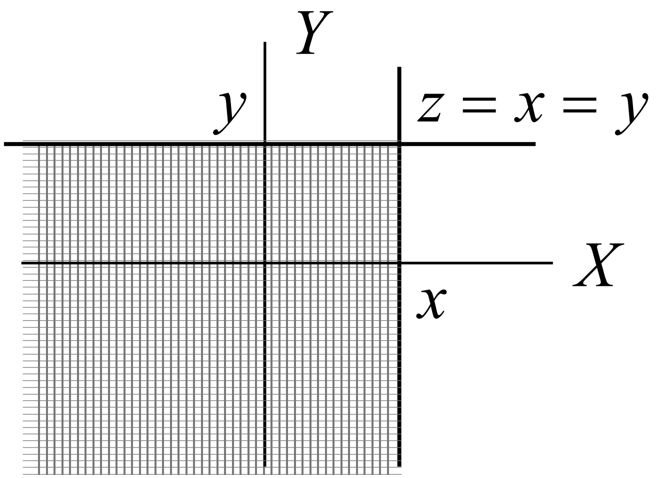
$$\Pr\{Z \le z\} = \Pr\{X \le z, Y \le z\}$$

If X and Y are independent,

$$\Pr\{Z \le z\} = \Pr\{X \le z\}\Pr\{Y \le z\}$$

$\therefore\ F_Z(z) = F_X(z)F_Y(z)$ and

$$f_Z(z) = f_X(z)F_Y(z) + f_Y(z)F_X(z)$$



Probability Density Functions of Two R.V.s

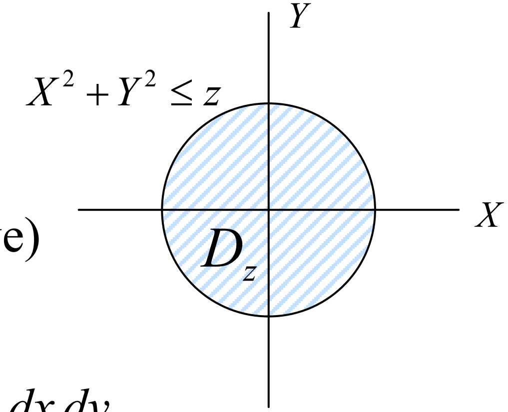
$$Z = X^2 + Y^2$$

If $z < 0$,

$F_Z(z) = 0$ (because, $X^2 + Y^2$ can never be negative)

If $z > 0$,

$$F_Z(z) = \Pr\{Z \le z\} = \Pr\{X^2 + Y^2 \le z\} = \iint_{x^2+y^2 \le z} f_{X,Y}(x,y)\, dx\, dy$$



Example:

Given $f_{X,Y}(x,y) = \dfrac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$, find pdf for $Z = X^2 + Y^2$.

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then, $dxdy = rdrd\theta$.

$$F_Z(z) = \int_0^{2\pi}\int_0^{\sqrt{z}} \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} rdrd\theta = \int_0^{\sqrt{z}} \frac{2\pi r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} dr = 1 - \exp\left(-\frac{z}{2\sigma^2}\right), \quad z > 0$$

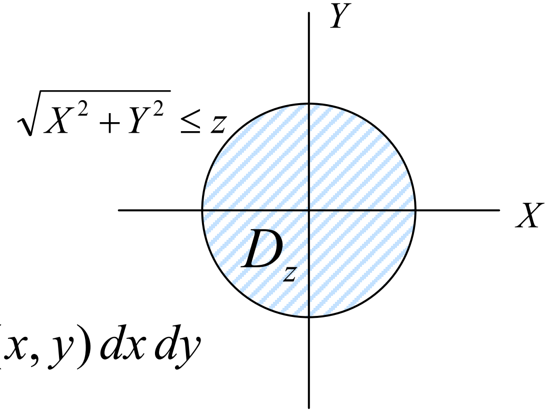
$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{1}{2\sigma^2}\exp\left(-\frac{z}{2\sigma^2}\right) u(z)$$

Probability Density Functions of Two R.V.s

$Z = \sqrt{X^2 + Y^2}$

If $z < 0$, $F_Z(z) = 0$

If $z > 0$,



$$F_Z(z) = \Pr\{Z \le z\} = \Pr\{\sqrt{X^2 + Y^2} \le z\} = \iint_{\sqrt{x^2+y^2} \le z} f_{X,Y}(x, y)\,dx\,dy$$

Example:

Given $f_{X,Y}(x, y) = \dfrac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$, find pdf for $Z = \sqrt{X^2 + Y^2}$.

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then, $dxdy = rdrd\theta$.

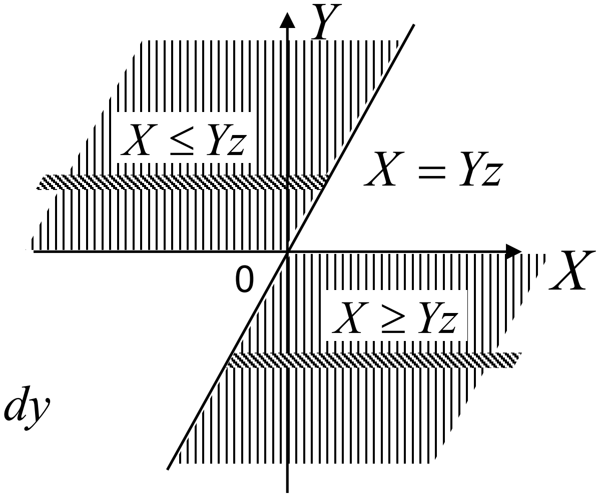
$$F_Z(z) = \int_0^{2\pi}\int_0^z \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} rdrd\theta = \int_0^z \frac{2\pi r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} dr = 1 - \exp\left(-\frac{z^2}{2\sigma^2}\right), \quad z > 0$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{z}{\sigma^2}\exp\left(-\frac{z^2}{2\sigma^2}\right)u(z)$$

Probability Density Functions of Two R.V.s

$$Z = X / Y$$

Given z, the function $x=yz$ is a line going through the origin.



$$F_Z(z) = \Pr\{Z \le z\} = \Pr\left\{\frac{X}{Y} \le z\right\} = \iint_{\text{shaded area}} f_{X,Y}(x,y)\,dx\,dy$$

because if $y > 0$, then $x \le yz$; if $y < 0$, then $x \ge yz$

$$F_Z(z) = \int_0^\infty \int_{-\infty}^{yz} f_{X,Y}(x,y)\,dx\,dy + \int_{-\infty}^0 \int_{yz}^\infty f_{X,Y}(x,y)\,dx\,dy$$

$$f_Z(z) = \int_0^\infty y\, f_{X,Y}(yz,y)\,dy - \int_{-\infty}^0 y f_{X,Y}(yz,y)\,dy = \int_{-\infty}^\infty |\,y\,| f_{X,Y}(yz,y)\,dy$$

Transformation of Multiple Random Variables

$Z=\varphi_1(X,Y)$ and $W=\varphi_2(X,Y)$ are two functions of r.v. X and Y.

Both φ_1 and φ_2 are continuous functions with corresponding inverse functions $X=\psi_1(Z,W)$ and $Y=\psi_2(Z,W)$, respectively.

Since all the events that map to $\{x_1 < X(\xi) \le x_2, y_1 < Y(\xi) \le y_2\}$ would also map to $\{z_1 < Z(\xi) \le z_2, w_1 < W(\xi) \le w_2\}$, we have

$$\Pr\{x_1 < X \le x_2, y_1 < Y \le y_2\} = \Pr\{z_1 < Z \le z_2, w_1 < W \le w_2\}.$$

or $$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dydx = \int_{z_1}^{z_2}\int_{w_1}^{w_2} f_{Z,W}(z,w)\,dwdz$$

But $$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dydx = \int_{z_1}^{z_2}\int_{w_1}^{w_2} f_{X,Y}(\psi_1(z,w),\psi_2(z,w))\,|J|\,dwdz$$

by way of change of variables with

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{\partial \psi_1}{\partial z} & \frac{\partial \psi_1}{\partial w} \\ \frac{\partial \psi_2}{\partial z} & \frac{\partial \psi_2}{\partial w} \end{vmatrix}$$

J is the Jacobian that relates the incremental area $dzdw$ to $dxdy$.

Transformation of Multiple R.V. (cont'd)

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dydx = \int_{z_1}^{z_2}\int_{w_1}^{w_2} f_{X,Y}(\psi_1(z,w),\psi_2(z,w))|J|\,dwdz$$

$$= \int_{z_1}^{z_2}\int_{w_1}^{w_2} f_{Z,W}(z,w)\,dwdz$$

Therefore, $f_{Z,W}(z,w) = |J|\, f_{X,Y}[\psi_1(z,w),\psi_2(z,w)]$

Example: $Z = XY, \quad W = X \;\Rightarrow\; X = W, \quad Y = Z/W$

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ w^{-1} & -\frac{z}{w^2} \end{vmatrix} = -w^{-1}$$

Thus, $f_{Z,W}(z,w) = \frac{1}{|w|} f_{X,Y}\left(w, \frac{z}{w}\right)$

The mariginals, $f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|w|} f_{X,Y}\left(w,\frac{z}{w}\right)dw$

$$f_W(w) = \int_{-\infty}^{\infty} \frac{1}{|w|} f_{X,Y}\left(w,\frac{z}{w}\right)dz$$

Transformation of Multiple R.V. - Example

Two random variables X and Y have a joint probability density function of the form

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 \le x, y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

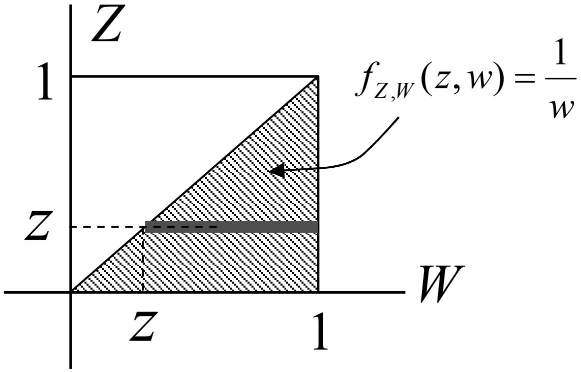
Find the pdf of $Z = XY$.

Use the result of the previous example:

$$Z = XY, \quad W = X \;\Rightarrow\; X = W, \quad Y = Z/W$$

$$f_{Z,W}(z,w) = \frac{1}{|w|} f_{X,Y}\left(w, \frac{z}{w}\right) = \frac{1}{w} \qquad 0 \le w \le 1 \text{ and } 0 \le z \le w$$

$$f_Z(z) = \int_z^1 \frac{1}{w}\,dw = \ln w \Big|_{w=z}^{1} = -\ln(z)$$



Transformation of Multiple R.V. - Example

Let $Z = aX + bY$ Then, $X = (dZ - bW)/(ad - bc)$

$W = cX + dY$ $Y = (-cZ + aW)/(ad - bc)$

assume $ad - bc = A \neq 0$

The Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} d/A & -b/A \\ -c/A & a/A \end{vmatrix} = \frac{1}{A}$$

$$f_{Z,W}(z,w) = \frac{1}{|ad - bc|} f_{X,Y}\left(\frac{dz - bw}{ad - bc}, \frac{-cz + aw}{ad - bc}\right)$$

But, often it's not as complicated as it seems.

$$E[ZW] = E[acX^2 + bdY^2 + (ad + bc)XY] = ac\overline{X^2} + bd\overline{Y^2} + (ad + bc)\overline{XY}$$

$$E[Z^2] = E[a^2X^2 + b^2Y^2 + 2abXY] = a^2\overline{X^2} + b^2\overline{Y^2} + 2ab\overline{XY}$$

First and Second Order Moments

If we are only to find the first and second order moment of a function of several random variables, we may not need to find the joint density function. Consider $X = \sum_{i=1}^{N} a_i X_i,$, a weighted sum of several r.v.s.

$$\overline{X} = E[X] = E\left[\sum_{i=1}^{N} a_i X_i\right] = \sum_{i=1}^{N} a_i E[X_i] = \sum_{i=1}^{N} a_i \overline{X_i}, \qquad X - \overline{X} = \sum_{i=1}^{N} a_i X_i - \sum_{i=1}^{N} a_i \overline{X_i} = \sum_{i=1}^{N} a_i (X_i - \overline{X_i})$$

$$\sigma_X^2 = E[(X - \overline{X})^2] = E\left[\sum_{i=1}^{N} a_i (X_i - \overline{X_i}) \sum_{j=1}^{N} a_j (X_j - \overline{X_j})\right]$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j E\left[(X_i - \overline{X_i})(X_j - \overline{X_j})\right] = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j C_{X_i X_j}$$

where $C_{X_i X_j} = E\left[(X_i - \overline{X_i})(X_j - \overline{X_j})\right]$ is the covariance between X_i and X_i

If these r.v.s are uncorrelated, $C_{X_i X_j} = \begin{cases} 0, & i \neq j \\ \sigma_{X_i}^2, & i = j \end{cases}$ Then, $\sigma_X^2 = \sum_{i=1}^{N} a_i^2 \sigma_{X_i}^2$

The variance of a weighted sum of uncorrelated random variables equals the weighted sum of the variances of the random variables (squared weights).